Block 13 - Capacitor Circuits and Energy

Student Group

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Block 13 - Capacitor Circuits and Energy

13.0 Intro

13.0.1 Learning Objectives

> After this 90-minute block, you can
>
> - identify series vs. parallel connections of capacitors from a circuit diagram,
> - compute equivalent capacitance $C_{\rm eq}$ for series and parallelnetworks,
> - use the key sharing rules: in **series** $Q_k=\text{const.}$ and voltages divide; in **parallel** $U_k=\text{const.}$ and charges divide,
> - apply the capacitor divider relation (two series capacitors),
> - determine stored energy, including a dimensional check to $\rm J$.

13.0.2 Preparation at Home

Well, again

- read through the present chapter and write down anything you did not understand.
- Also here, there are some clips for more clarification under 'Embedded resources' (check the text above/below, sometimes only part of the clip is interesting).

For checking your understanding please do the following exercises:

- 5.9.5

13.0.3 90-minute plan

1. Warm-up (10 min):
    1. Quick quiz (2–3 items): series or parallel? which rule applies (constant U or constant Q)?
    2. Recall $Q=C\,U$ and energy $W=\tfrac12 C U^2$ (units).
2. Core concepts & derivations (35 min):
    1. Derive $C_{\rm eq}$ for **series** from Kirchhoff's voltage law and $Q=\text{const.}$; derive voltage division $U_k=\dfrac{Q}{C_k}$.
    2. Derive $C_{\rm eq}$ for **parallel** from Kirchhoff's current/charge balance and $U=\text{const.}$; obtain $Q_k=C_k U$.
    3. Energy in the electric field: integrate $dW=U\,dq$ → $W=\tfrac12 C U^2$; short dimensional check.
3. Practice (35 min):
    1. Two short worked examples: mixed series/parallel network; two-capacitor divider with given U (find U_1, U_2, W on each).
    2. Short simulation tasks (use the two embedded Falstad circuits in this page): observe U_k, Q_k when toggling the switch or changing values.
    3. Mini-problems: "double a plate area / halve distance" reasoning on C and W.
4. Wrap-up (10 min):
    1. Common-pitfalls checklist and one exit-ticket calculation.

13.0.4 Conceptual overview

1. **What stays the same?** In **series** all capacitors carry the **same charge** Q; in **parallel** all capacitors see the **same voltage** U.
2. **How do totals form?** Capacitances **add inversely** in series and **add directly** in parallel. This mirrors resistors but with the roles swapped.
3. **Voltage/charge sharing:** In series, the **smaller** C_k takes the **larger** U_k ($U_k=Q/C_k$). In parallel, the **larger** C_k takes the **larger** Q_k ($Q_k=C_k U$).
4. **Energy viewpoint:** Charging needs work against the field; $W=\tfrac12 C U^2=\tfrac12 Q U=\dfrac{Q^2}{2C}$. Dimensional check: $[C]=\rm F=\dfrac{A\,s}{V}$, so $[C U^2]=\dfrac{A\,s}{V}\,V^2=A\,s\,V=J$.
5. **Design intuition:** Increasing plate area A or dielectric ε_r raises C and thus stored W at the same U; increasing gap d lowers C.

13.1 Core content

13.1.1 Series Circuit of Capacitor

If capacitors are connected in series, the charging current I into the individual capacitors $C_1 ... C_n$ is equal. Thus, the charges absorbed ΔQ are also equal: \begin{align*} \Delta Q = \Delta Q_1 = \Delta Q_2 = ... = \Delta Q_n \end{align*}

Furthermore, after charging, a voltage is formed across the series circuit, which corresponds to the source voltage U_q. This results from the addition of partial voltages across the individual capacitors. \begin{align*} U_q = U_1 + U_2 + ... + U_n = \sum_{k=1}^n U_k \end{align*}

It holds for the voltage $U_k = \Large{{Q_k}\over{C_k}}$.
If all capacitors are initially discharged, then $U_k = \Large{{\Delta Q}\over{C_k}}$ holds. Thus \begin{align*} U_q &= &U_1 &+ &U_2 &+ &... &+ &U_n &= \sum_{k=1}^n U_k \\ U_q &= &{{\Delta Q}\over{C_1}} &+ &{{\Delta Q}\over{C_2}} &+ &... &+ &{{\Delta Q}\over{C_3}} &= \sum_{k=1}^n {{1}\over{C_k}}\cdot \Delta Q \\ {{1}\over{C_{ \rm eq}}}\cdot \Delta Q &= &&&&&&&&\sum_{k=1}^n {{1}\over{C_k}}\cdot \Delta Q \end{align*}

Thus, for the series connection of capacitors $C_1 ... C_n$:

\begin{align*} \boxed{ {{1}\over{C_{ \rm eq}}} = \sum_{k=1}^n {{1}\over{C_k}} } \end{align*} \begin{align*} \boxed{ \Delta Q_k = {\rm const.}} \end{align*}

For initially uncharged capacitors, (voltage divider for capacitors) holds: \begin{align*} \boxed{Q = Q_k} \end{align*} \begin{align*} \boxed{U_{ \rm eq} \cdot C_{ \rm eq} = U_{k} \cdot C_{k} } \end{align*}

In the simulation below, besides the parallel connected capacitors C_1, C_2,C_3, an ideal voltage source U_q, a resistor R, a switch S, and a lamp are installed.

- The switch S allows the voltage source to charge the capacitors.
- The resistor R is necessary because the simulation cannot represent instantaneous charging. The resistor limits the charging current to a maximum value.

- The capacitors can be discharged again via the lamp.

13.1.2 Parallel Circuit of Capacitors

If capacitors are connected in parallel, the voltage U across the individual capacitors $C_1 ... C_n$ is equal. It is therefore valid:

\begin{align*} U_q = U_1 = U_2 = ... = U_n \end{align*}

Furthermore, during charging, the total charge ΔQ from the source is distributed to the individual capacitors. This gives the following for the individual charges absorbed: \begin{align*} \Delta Q = \Delta Q_1 + \Delta Q_2 + ... + \Delta Q_n = \sum_{k=1}^n \Delta Q_k \end{align*}

If all capacitors are initially discharged, then $Q_k = \Delta Q_k = C_k \cdot U$
Thus \begin{align*} \Delta Q &= & Q_1 &+ & Q_2 &+ &... &+ & Q_n &= \sum_{k=1}^n Q_k \\ \Delta Q &= &C_1 \cdot U &+ &C_2 \cdot U &+ &... &+ &C_n \cdot U &= \sum_{k=1}^n C_k \cdot U \\ C_{ \rm eq} \cdot U &= &&&&&&&& \sum_{k=1}^n C_k \cdot U \\ \end{align*}

Thus, for the parallel connection of capacitors $C_1 ... C_n$:

\begin{align*} \boxed{ C_{ \rm eq} = \sum_{k=1}^n C_k } \end{align*} \begin{align*} \boxed{ U_k = {\rm const.}} \end{align*}

For initially uncharged capacitors, (charge divider for capacitors) holds: \begin{align*} \boxed{\Delta Q = \sum_{k=1}^n Q_k} \end{align*}

\begin{align*} \boxed{ {{Q_k}\over{C_k}} = {{\Delta Q}\over{C_{ \rm eq}}} } \end{align*}

In the simulation below, again, besides the parallel connected capacitors C_1, C_2,C_3, an ideal voltage source U_q, a resistor R, a switch S, and a lamp are installed.

13.1.3 Energy in the electric Field

Now we want to see how much energy is stored in a capacitor during charging. When we want to charge a capacior charges have be separated (see figure 1). This gets more and more difficult as more charges were moved, since these already moved charges create an electric field.

Fig. 1: summary of electrostatics

We already had a first look onto the energy in the electric field in block09.
There, we got:

$$\begin{align*} \Delta W &= \int \vec{F} d\vec{r} \\ &= q \int \vec{E} d\vec{r} \\ &= q \cdot U \\ dW &= dq \cdot U \end{align*}$$

Now, For a capacitor we include the formula for the capacitor $C = {{q}\over{U}}$, or better its rearranged version $U = {{q}\over{C}}$:

$$\begin{align*} dW &= dq \cdot {{q}\over{C}} \\ \int dW &= \int {{q}\over{C}} dq \end{align*}$$

Here we again see, that the needed energy portion dW to move a portion dq is also related to the already moved charges q.
To get the energy $Delta W$ needed to move all of the charges $Q = \int dq$ we have to integrate from 0 to Q:

$$\begin{align*} \Delta W &= \int_0^Q dW \\ &= \int_0^Q {{q}\over{C}} dq \\ &= {{1}\over{2}}{{Q^2}\over{C}} \\ \end{align*}$$
$$\begin{align*} \boxed{ \Delta W = {{1}\over{2}}{{Q^2}\over{C}} = {{1}\over{2}}QU = {{1}\over{2}}CQ^2 } \end{align*}$$

13.2 Common pitfalls

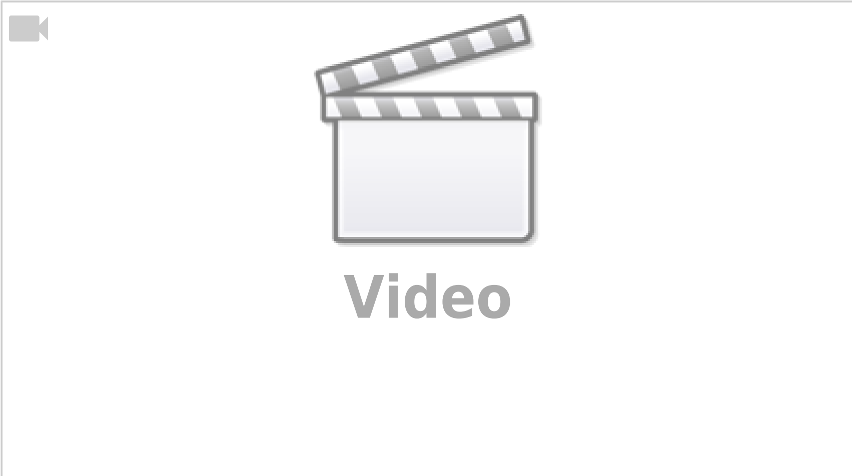
- Mixing up the rules: writing $C_{\rm eq}=C_1+C_2$ for **series** (wrong) or $\dfrac{1}{C_{\rm eq}}=\dfrac{1}{C_1}+\dfrac{1}{C_2}$ for **parallel** (wrong).
- Forgetting which quantity is equal: **series $\Rightarrow Q_k=\text{const.}$**, **parallel $\Rightarrow U_k=\text{const.}$**.
- Applying the **resistive** voltage divider $U_1=\dfrac{R_1}{R_1+R_2}U$ to capacitors. For capacitors in series it inverts: $U_1=\dfrac{C_2}{C_1+C_2}U$.
- Ignoring **initial charge states**: pre-charged capacitors reconnected will redistribute charge; use charge conservation on isolated nodes before using $Q=C\,U$.
- Dropping units or mixing forms of energy: always keep $W=\tfrac12 C U^2=\tfrac12 Q U=\dfrac{Q^2}{2C}$ and check $\rm J$.

13.3 Exercises

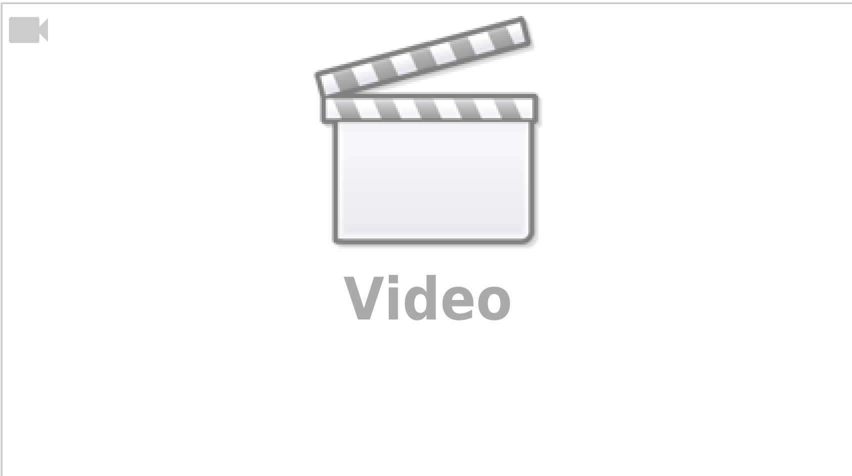
Task 5.8.1 Calculating a circuit of different capacitors

See https://www.youtube.com/watch?v=vSeSHAmpd4Y

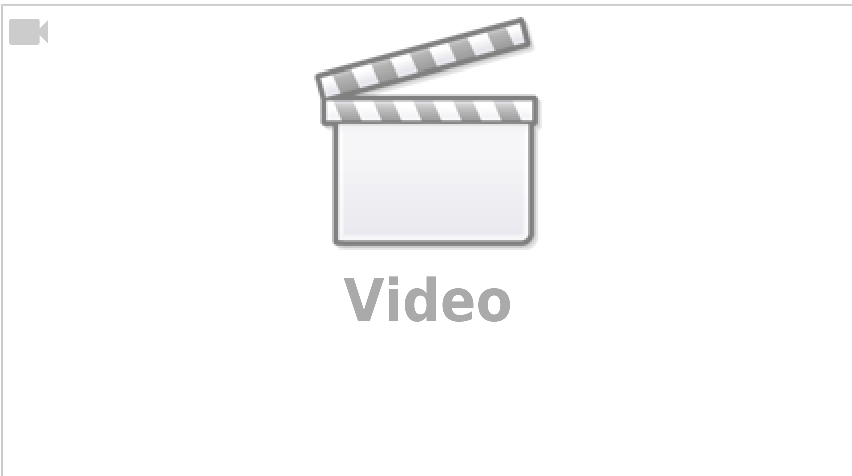
Task 5.9.1 Layered Capacitor Task



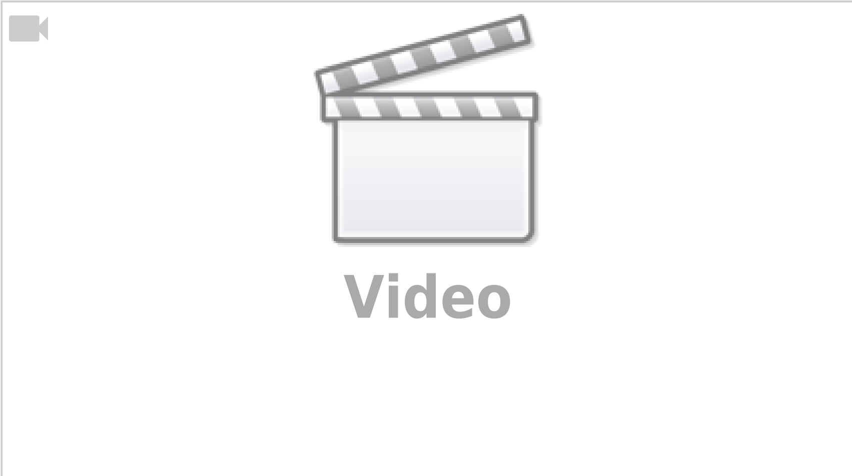
Exercise 5.9.2 Further capacitor charging/discharging practice Exercise



Exercise 5.9.3 Further practice charging the capacitor

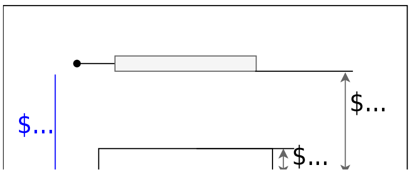


Exercise 5.9.4 Charge balance of two capacitors



Exercise 5.9.5 Capacitor with glass plate

Fig. 2: Structure of a capacitor with glass plate



Two parallel capacitor plates face each other with a distance $d_{ \rm K} = 10~{ \rm mm}$. A voltage of $U = 3'000~{ \rm V}$ is applied to the capacitor. Parallel to the capacitor plates, there is a glass plate ($\varepsilon_{ \rm r, G}=8$) with a thickness $d_{ \rm G} = 3~{ \rm mm}$ in the capacitor.

1. Calculate the partial voltages $U_{ \rm G}$ in the glass and $U_{ \rm A}$ in the air gap.
2. What is the maximum thickness of the glass pane if the electric field $E_{ \rm 0, G} =12 ~{ \rm kV/cm}$ must not exceed?

Exercise E5 Capacitor
(written test, approx. 12 % of a 120-minute written test, SS2024)

[illegible]

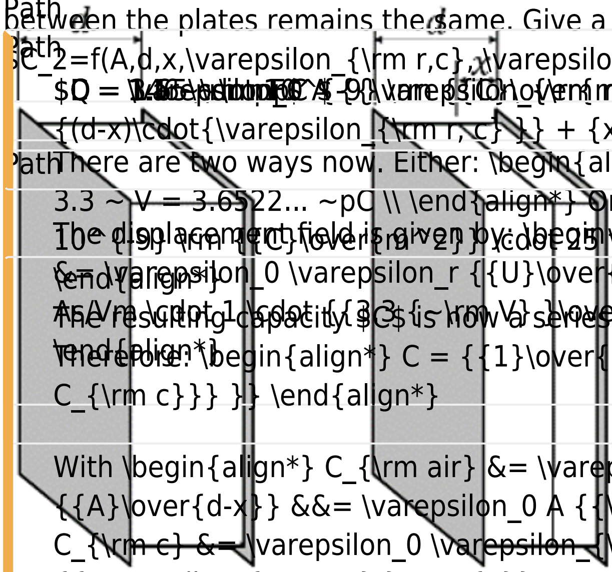
Result

right side image.

The contaminant has $\varepsilon_{\rm r,c}>\varepsilon_{\rm r,air}$, while the distance between the plates remains the same. Give a generalized formula $C_2=f(A,d,x,\varepsilon_{\rm r,c},\varepsilon_{\rm r,air})$.

Path

Path



$C = \varepsilon_0 A {{\varepsilon_{\rm r, air}}\cdot \varepsilon_{\rm r, c} }\over{ {(d-x)\cdot{\varepsilon_{\rm r, c} }} + {x\cdot{ \varepsilon_{\rm r, air}}} }}$

Path

There are two ways now. Either: \begin{align*} Q &= C \cdot U = 1.1 ~\rm pF \cdot 3.3 ~ V = 3.6522... ~pC \\ \end{align*} Or: \begin{align*} Q &= D \cdot A = 146 \cdot 10^{-9} \rm {{C}\over{m^2}} \cdot 25 \cdot 10^{-6} ~m^2 = 3.6522... ~pC \\ \end{align*}

The displacement field is given by: \begin{align*} D &= \varepsilon_0 \varepsilon_r E \\ &= \varepsilon_0 \varepsilon_r {{U}\over{d}} \\ &= 8.854 \cdot 10^{-12} ~\rm As/Vm \cdot 1 \cdot {{3.3 {~\rm V} }\over{200 \cdot 10^{-6} {~\rm m} }} \\ \end{align*}

The resulting capacity C is now a series circuit of $C_{\rm air}$ and $C_{\rm c}$. Therefore: \begin{align*} C = {{1}\over{ {{1}\over{C_{\rm air} }} + {{1}\over{ C_{\rm c}}} }} \end{align*}

With \begin{align*} C_{\rm air} &= \varepsilon_0 \varepsilon_{\rm r, air} {{A}\over{d-x}} &&= \varepsilon_0 A {{\varepsilon_{\rm r, air}}\over{d-x}} \\ C_{\rm c} &= \varepsilon_0 \varepsilon_{\rm r, c} {{A}\over{x}} &&= \varepsilon_0 A {{\varepsilon_{\rm r, c} }\over{x}} \\ \end{align*}

This leads to: \begin{align*} C = \varepsilon_0 A {{1}\over{ {{d-x}\over{\varepsilon_{\rm r, air} }} + {{x}\over{ \varepsilon_{\rm r, c}}} }} \end{align*}

In the following such a sensor is given with:

- Plate area : $A=25 ~\rm mm^2$
- Distance between both plates: $d=200 ~\rm \mu m$
- Air between the plates: $\varepsilon_{\rm r,air}=1$
- Supply voltage: $3.3 ~\rm V$
- Boundary effects on the end of the layers shall be ignored in the following calculations.

$\varepsilon_{0} =8.854 \cdot 10^{-12} ~\rm F/m $

1. Calculate the capacity C.

Path

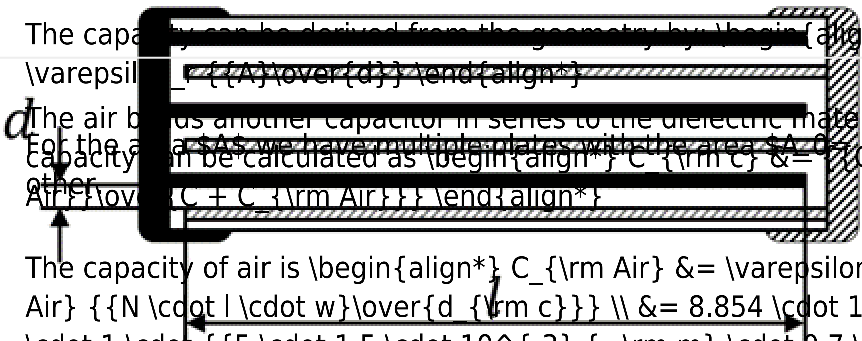
\begin{align*} C &= \varepsilon_0 \varepsilon_r {{A}\over{d}} \\ &= 8.854 \cdot 10^{-12} ~\rm As/Vm \cdot 1 \cdot {{25 \cdot 10^{-6} {~\rm m} }\over{200 \cdot 10^{-6} {~\rm m} }} \end{align*}

Exercise E1 Capacitor

(written test, approx. 7 % of a 120-minute written test, SS2022)

Given is the multilayer capacitor shown below, with the following dimensions:

- Length of layer overlap: $l=1.5 ~\rm mm$
- Distance between single layers: $d=1.0 ~\rm \mu m$
- Depth of component: $w=0.7 ~\rm mm$
- Number of layers (shown in the picture): 3 left-side and 3 right-side layers.



The material shall have a dielectric permittivity of $\varepsilon_r=3$.

The following calculations shall ignore boundary effects on the end of the layers.

$\varepsilon_0 = 8.854 \cdot 10^{-12} ~\rm As/Vm$

1. What is the field strength in the dielectric material between the layer, when a voltage of $U=6.3 ~\rm V$ is applied?

Path

The electric field strength E is given by: \begin{align*} E &= {{U}\over{d}} \\ &= {{6.3 ~\rm V}\over{1 \cdot 10^{-6} ~\rm m}} \\ \end{align*}

2. Calculate the capacity C.

Path

The capacity can be derived from the geometry by: \begin{align*} C = \varepsilon_0 \varepsilon_r {{A}\over{d}} \end{align*}

For the area A we have multiple plates with the area $A_0 = l \cdot w$ facing each other.

How many "multiple plates" do we have to consider?

For this, we have to count facing areas A_0. There are $N=5$.

3. Due to production problems the left-side layers are covered with a layer of $d_{\rm c}=0.1 ~\rm \mu m$ of air ($\varepsilon_{r, \rm c}=1$), while the thickness of the dielectric material remains the same. What is the new capacity $C_{\rm c}$?

Result

Path

$C_{\rm c} = 0.107 ~\rm nF$

Path

The air builds another capacitor in series to the dielectric material. Therefore, the capacity can be calculated as \begin{align*} C_{\rm c} &= {{C \cdot C_{\rm Air}}\over{C + C_{\rm Air}}} \end{align*}

The capacity of air is \begin{align*} C_{\rm Air} &= \varepsilon_0 \varepsilon_{r,\rm Air} {{N \cdot l \cdot w}\over{d_{\rm c}}} \\ &= 8.854 \cdot 10^{-12} ~\rm As/Vm \cdot 1 \cdot {{5 \cdot 1.5 \cdot 10^{-3} {~\rm m} \cdot 0.7 \cdot 10^{-3} {~\rm m} }\over{0.1 \cdot 10^{-6} {~\rm m} }} \\ &= 0.465... ~\rm nF \end{align*}

By this the overall capacity is \begin{align*} C_{\rm c} &= {{0.139... ~\rm nF \cdot 0.465... ~\rm nF}\over{0.139... ~\rm nF + 0.465... ~\rm nF}} \end{align*}

> Therefore, the formula is \begin{align*} C &= \varepsilon_0 \varepsilon_r {{N \cdot l \cdot w}\over{d}} \\ &= 8.854 \cdot 10^{-12} ~\rm As/Vm \cdot 3 \cdot {{5 \cdot 1.5 \cdot 10^{-3} {~\rm m} \cdot 0.7 \cdot 10^{-3} {~\rm m} }\over{1 \cdot 10^{-6} {~\rm m} }} \end{align*}

Embedded resources

The equivalent capacitor for series of parallel configuration is well explained here