6 Introduction to Alternating Current Technology

Student Group

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Table of Contents

6 Introduction to Alternating Current Technology

Up to now, we had analyzed DC signals (chapters 1. - 4.) and abrupt voltage changes for (dis)charging capacitors (chapter 5.). In households, we use alternating voltage (AC) instead of a constant voltage (DC). This is due to at least three main facts

1. Often the voltage given by the **power plant is AC**. This is true for example in all power plants which use electric generators. In these, the mechanical energy of a rotating system is transformed into electric energy using moving magnets, which induce an alternating electric voltage. Some modern plants, like photovoltaic plants, do not primarily generate AC voltages.
2. For long-range power transfer the power losses $P_{\rm loss}$ can be reduced by reducing the currents I since $P_{\rm loss}=R\cdot I^2$. Therefore, for constant power transfer, the voltage has to be increased. This is much easier done with AC voltages: **AC enables the transformation of a lower voltage to a higher** by the use of alternating magnetic fields in a transformer.
3. AC signals have **at least one more value** which can be used for understanding the situation of the source or load. This simplifies the power and load management in a complex power network.

This does not mean that DC power lines are useless or only full of disadvantages:

- A lot of modern loads need DC voltages, like battery-based systems (laptops, electric cars, smartphones). Others can simply be changed into DC loads like systems with electric motors (refrigerators, ovens, lighting, heating).
- Long-range power transfer with DC voltages show often much lower power losses.

Besides the applications in power systems AC values are also important in communication engineering. Acoustic and visual signals like sound and images can often be considered as wavelike AC signals. Additionally, also for signal transfer like Bluetooth, RFID, and antenna design AC signals are important.

To understand these systems a bit more, we will start this chapter with a first introduction to AC systems.

> If you have trouble understanding the complex numbers please refer to the following videos:
>
> - The question “What's the point of complex numbers?” is addressed here and here
> - How to calculate with complex numbers (sum, difference, product) can be seen in this video
> - The geometric interpretation of the complex multiplication or: “Why is the amount and angle to be added in the multiplication?” is shown here, and here

6.1 Description of time-dependent Signals

Learning Objectives

By the end of this section, you will be able to:

1. know which types of time-dependent waveforms there are and be able to assign them
2. Know the relationship between amplitude and peak-to-peak value.
3. Know the relationship between period, frequency, and angular frequency.
4. Know the difference between zero phase angle and phase shift angle.
5. Know the direction of the phase shift angle.
6. know the formula symbols of the above-mentioned quantities.

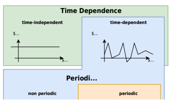
6.1.1 Description of Classification of time-dependent Signals

Voltages and currents in the following chapters will be time-dependent values. As already used in chapter 5. for the time-dependent values lowercase letters will be written.

By these time-dependent values, any temporal form of the voltage/current curves is possible (see figure 1).

- We distinguish periodic and non-periodic signals
- One important family of periodic signals is sinusoidal signals
- Sinusoidal signals can be mixed with DC signals

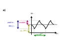
Fig. 1: Classification of time-dependent values



In the following, we will investigate mainly pure AC signals.

6.1.2 Descriptive Values of AC Signals

Fig. 2: Values of AC signals



There are some important characteristic values when investigating AC signals (figure 2). For the signal itself, these are:

- The **DC voltage** or DC offset is given by the value $U_{\rm DC}$ of $V_{\rm DC}$ (in German: Gleichanteil). The DC component also defines the average value of an AC signal.
- The maximum deviation from the DC value is called **peak voltage** $U_\rm p$ (in German : *Spitzespannung*). Specifically for sinusoidal signals the **peak voltage** $U_\rm p$ is also called **amplitude** $\hat{U}$ (in German: *Scheitelwert* or *Amplitude*).
- The voltage difference between maximum and minimum deviation is called **peak-to-peak voltage** $U_{\rm pp}$ (in German: *Spitze-Spitze-Spannung*).
  Be aware, that in English texts the term amplitude is also often used for (non-sinusoidal) $U_{\rm pp}$. Based on German DIN standards the term amplitude is only valid for the sinusoidal peak voltage.

Additionally, there are also characteristic values related to time:

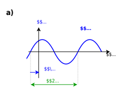
- The shortest time difference for the signal to repeat is called **period** T.
- Based on the period T the frequency $f = {{1}\over{T}}$ can be derived. The unit of the frequency is $1 ~{\rm Hz} = 1 ~\rm Hertz$.
- For calculation, often the **angular frequency** ω is used. The angular frequency is given by $\omega = {{2\pi}\over{T}}$ with the unit ${{1}\over{s}}$.
  The angular frequency represents the angle that is covered in one second.
- Another handy value is the time offset between the start of the sinus wave ($u(t)=0~\rm V$ and rising) and $t=0 ~\rm s$. This difference is often written based on an angular difference and is called the **phase angle** or **initial phase** φ_U (in German: *Nullphasenwinkel*). This then has to be calculated back to a time value: $\Delta t= {{\varphi_U}\over{\omega}}= \varphi_U\cdot{{T}\over{2\pi}}$

Mathematically, the AC voltages and currents can be written as: $$u(t)=\hat{U}\cdot \sin(\omega t + \varphi_U)$$ $$i(t)=\hat{I}\cdot \sin(\omega t + \varphi_I)$$

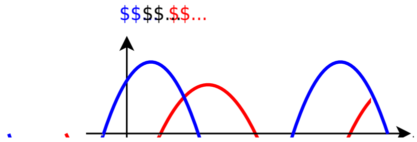
Between the AC voltages and currents, there is also another important characteristic: The **phase difference** $\Delta \varphi$ is given by $\Delta \varphi = \varphi_U - \varphi_I$. The phase difference shows how far the momentary value of the current is ahead of the momentary value of the voltage.

Notice:

The initial phase φ_0 has a direction/sign which has to be considered. In the case **a)** in the picture the zero-crossing of the sinusoidal signal is before $t=0$ or $\omega t =0$. Therefore, the initial phase φ_0 is positive.



Similarly also for the phase difference $\Delta \varphi$ the direction has to be taken into account. In the following image, the zero-crossing of the voltage curve is before the zero-crossing of the current. This leads to a positive phase difference $\Delta \varphi$.



6.2 Averaging of AC Signals

Learning Objectives

By the end of this section, you will be able to:

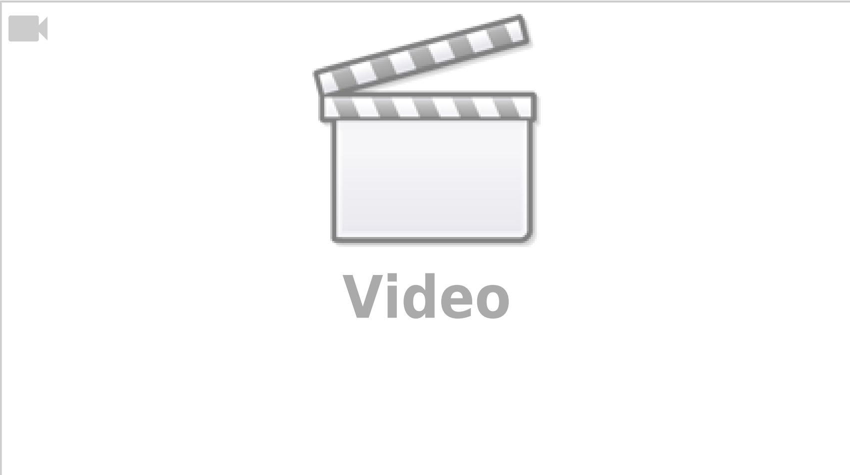
1. calculate the arithmetic mean, the rectified value, and the RMS value.
2. know these mean values for sinusoidal quantities.
3. know the reason for using the RMS value.

To analyze AC signals more, often different types of averages are taken into account. The most important values are:

1. the arithmetic mean $\overline{X}$
2. the rectified value $\overline{|X|}$

3. the RMS value X

These shall be discussed in the following. The video “Alternating Current AC Basics - Part 1” of EEVblog explains the ideas behind these values alternatively to the following subchapter:



6.2.1 The Arithmetic Mean

The arithmetic mean is given by the (equally weighted) averaging of the signed measuring points.
For finite values the arithmetic mean is given by: $$\overline{X}={{1}\over{n}}\cdot \sum_{i=1}^n x_i$$

For functions, it is given by: $$\boxed{\overline{X}={{1}\over{T}}\cdot \int_{t=t_0}^{t_0 + T} x(t) {\rm d}t}$$

For pure AC signals, the arithmetic mean is $\overline{X}=0$, since the unsigned value of the integral between the upper half-wave and 0 is equal to the unsigned value of the integral between the lower half-wave and 0.

6.2.2 The Rectified Value

Since the arithmetic mean of pure AC signals with $\overline{X}=0$ does not really give an insight into the signal, different other (weighted) averages can be used.
One of them is the rectified value. For this, the signal is first rectified (visually: negative values are folded up onto the x-axis) and then averaged.
For finite values, the rectified value is given by: $$\overline{|X|}={{1}\over{n}}\cdot \sum_{i=1}^n |x_i|$$

For functions, it is given by: $$\boxed{\overline{|X|}={{1}\over{T}}\cdot \int_{t=t_0}^{t_0 + T} |x(t)| {\rm d}t}$$

> For pure AC signals this results in:
>
> \begin{align*} \overline{|X|} &= {{1}\over{T}}\cdot \int_{t=t_0}^{t_0 + T} |\hat{X}\cdot \sin(\omega t + \varphi) | {\rm d}t \\ \end{align*}
>
> Without limiting the generality, we use $\varphi=0$ and $t_0 = 0$ \begin{align*} \overline{|X|} &= {{1}\over{T}}\cdot \int_{t=0 }^{T } |\hat{X}\cdot \sin(\omega t) | {\rm d}t \\ \end{align*}

> Since $sin(\omega t)\geq0$ for $t\in [0,\pi]$, the integral can be changed and the absolute value bars can be excluded like the following
> \begin{align*} \overline{|X|} &= {{1}\over{T}}\cdot 2 \cdot \int_{t=0}^{T/2} \hat{X}\cdot \sin({{2\pi}\over{T}} t) {\rm d}t \\ &= 2 \cdot {{1}\over{T}}\cdot [-\hat{X}\cdot {{T}\over{2\pi}}\cdot \cos({{2\pi}\over{T}} t)]_{t=0}^{T/2} \\ &= 2 \cdot {{1}\over{T}}\cdot {{T}\over{2\pi}}\cdot \hat{X}\cdot [-\cos({{2\pi}\over{T}} t)]_{t=0}^{T/2} \\ &= {{1}\over{\pi}}\cdot \hat{X} \cdot [1+1] \\ \boxed{\overline{|X|} = {{2}\over{\pi}}\cdot \hat{X} \approx 0.6366 \cdot \hat{X}}\\ \end{align*}

Exercise 6.2.1 The Rectified Value of rectangular and triangular signals

Calculate the rectified value of rectangular and triangular signals! Use similar symmetry simplifications as shown for AC signals. Compare it to the values shown in figure 3.

6.2.3 The RMS Value

Often it is important to be able to compare AC signals to DC signals by having equivalent values. But what does equivalent mean?
Most importantly, these “equivalent values” are used to compare the output power of a system. One of these equivalent values is the supply voltage value of $230~V$ (or in some countries $110V$). How do we come to these values?

We want to find the voltage U_{DC} and I_{DC} of a DC source, that the output power P_{DC} on a resistor R is similar to the output power P_{AC} of an AC source with the instantaneous values $u(t)$ and $i(t)$. For this, we have to consider the instantaneous power $p(t)$ for a distinct time t and integrate this over one period T.

\begin{align*} P_{\rm DC} &= P_{\rm AC} \\ U_{DC} \cdot I_{\rm DC} &= {{1}\over{T}} \int_{0}^{T} u(t) \cdot i(t) {\rm d}t \\ R \cdot I_{\rm DC}^2 &= {{1}\over{T}} \int_{0}^{T} R \cdot i^2(t) {\rm d}t \\ I_{\rm DC}^2 &= {{1}\over{T}} \int_{0}^{T} i^2(t) {\rm d}t \\ \rightarrow I_{\rm DC} &= \sqrt{{{1}\over{T}} \int_{0}^{T} i^2(t) {\rm d}t} \end{align*}

A similar approach can be used on instantaneous voltage $u(t)$. Generally, the RMS value of X is given by \begin{align*} \boxed{X_{\rm RMS} = \sqrt{{{1}\over{T}} \int_{0}^{T} x^2(t) {\rm d}t}} \end{align*} What is the meaning of RMS? Simple:

By this abbreviation, one can also not forget in which order the formula has to be written… Often the RMS value is also called effective value (in German: Effektivwert).

Note:

- The heat dissipation on a resistor R of an AC current with the RMS value of $I_{\rm RMS} = 1~\rm A$ is equal to the heat dissipation of a DC current with $I_{\rm DC} = 1~\rm A$.
- To shorten writing formulas, the values of AC signals given with uppercase letters will represent the RMS value in the following: $U = U_{\rm RMS}$, $I = I_{\rm RMS}$.
- It holds for AC signals and their RMS values:
    - The resistance is $R={{U}\over{I}}$
    - The power dissipation on a resistor is $P=U\cdot I$

For pure AC signals this results in:

$$\begin{align*} X &= \sqrt{{{1}\over{T}}\cdot \int_{t=t_0}^{t_0 + T} x^2(t) {\rm d}t} \\ &= \sqrt{{{1}\over{T}}\cdot \int_{t=0} ^{T} \hat{X}^2\cdot \sin^2(\omega t) {\rm d}t} \\ &= \sqrt{{{1}\over{T}}\cdot \int_{t=0} ^{T} \hat{X}^2\cdot {{1}\over{2}}\cdot (1- \cos(2\cdot \omega t)) {\rm d}t} \\ &= \sqrt{{{1}\over{T}}\cdot \hat{X}^2\cdot {{1}\over{2}}\cdot [t + {{1}\over{2\omega }}\cdot \sin(2\cdot \omega t)]_{0}^{T}} \\ &= \sqrt{{{1}\over{T}}\cdot \hat{X}^2\cdot {{1}\over{2}}\cdot (T - 0 + 0 - 0)} \\ &= \sqrt{{{1}\over{2}}\cdot \hat{X}^2} \\ \boxed{X = {{1}\over{\sqrt{2}}}\cdot \hat{X} \approx 0.707\cdot \hat{X}}\\ \end{align*}$$

This can also be seen in this youtube video.

Note:

In the following chapters, we will often use for a physical value $x(t)$ a dependency on $\sqrt{2}X$ instead of $\hat{X}$. Therefore, the sinusoidal formula of a physical value x will be : $x(t)=\hat{X}\cdot \sin(\omega t + \varphi_x) \rightarrow x(t)=\sqrt{2}{X}\cdot \sin(\omega t + \varphi_x) $

Exercise 6.3.2 The RMS Value of rectangular and triangular signals

Calculate the RMS value of rectangular and triangular signals! Use similar symmetry simplifications as shown for AC signals. Compare it to the values shown in figure 3.

6.2.4 Comparison of the different Averages

The following simulation shows the different values for averaging a rectangular, a sinusoidal, and a

triangular waveform.
Be aware that one has to wait for a full period to see the resulting values on the right outputs of the average generating blocks.

Fig. 3: The averages of different signals

6.3 AC Two-Terminal Networks

Learning Objectives

By the end of this section, you will be able to:

1. know that real, lossy components are described by equivalent circuits of ideal components.
2. know and be able to apply the definition of apparent resistance, apparent conductance, impedance, and admittance.

In the chapters 2. Simple Circuits and 3 Non-ideal Sources and Two-terminal Networks we already have seen, that it is possible to reduce complex circuitries down to equivalent resistors (and ideal sources). This we will try to adopt for AC components, too.

We want to analyze how the relationship between the current through a component and the voltage drop on this component behaves when an AC current is applied.

6.3.1 Resistance

We start with Ohm's law, which states, that the instantaneous voltage $u(t)$ is proportional to the instantaneous current $i(t)$ by the factor R. $$u(t) = R \cdot i(t)$$

Then we insert the functions representing the instantaneous signals: $x(t)= \sqrt{2}{X}\cdot \sin(\omega t + \varphi_x)$: $$\sqrt{2}{U}\cdot \sin(\omega t + \varphi_u) = R \cdot \sqrt{2}{I}\cdot \sin(\omega t + \varphi_i)$$

Since we know, that $u(t)$ must be proportional to $i(t)$ we conclude that for a resistor $\varphi_u=\varphi_i$!

\begin{align*} R &= {{\sqrt{2}{U}\cdot \sin(\omega t + \varphi_i)}\over{\sqrt{2}{I}\cdot \sin(\omega t + \varphi_i) }} \\ &= {{U}\over{I}} \end{align*}

Fig. 4: time course of instantaneous voltage and current on a resistance

This was not too hard and quite obvious. But, what about the other types of passive two-terminal networks - namely the capacitance and inductance?

6.3.2 Capacitance

For the capacitance we have the basic formula: $$C={{Q}\over{U}}$$ This formula is also true for the instantaneous values: $$C={{q(t)}\over{u(t)}}$$ Additionally, we know, that the instantaneous current is defined by $i(t)={{{\rm d}q(t)}\over{{\rm d}t}}$.

By this we can set up the formula: \begin{align*} i(t) &= {{{\rm d}q(t)}\over{{\rm d}t}} \\ &= {{\rm d}\over{{\rm d}t}}\left(C \cdot u(t) \right) \end{align*}

Now, we insert the functions representing the instantaneous signals and calculate the derivative: \begin{align*} \sqrt{2}{I}\cdot \sin(\omega t + \varphi_i) &= {{\rm d}\over{{\rm d}t}}\left(C \cdot \sqrt{2}{U}\cdot \sin(\omega t + \varphi_u) \right) \\ &= C \cdot \sqrt{2}{U}\cdot \omega \cdot \cos(\omega t + \varphi_u) \\ \\ {I}\cdot \sin(\omega t + \varphi_i) &= C \cdot {U}\cdot \omega \cdot \sin(\omega t + \varphi_u + {{1}\over{2}}\pi) \tag{6.3.1} \end{align*}

Equating coefficients in $(6.3.1)$ leads to: \begin{align*} I &= C \cdot U \cdot \omega \\ {{U}\over{I}} &= {{1}\over{\omega \cdot C}} \end{align*} and: \begin{align*} \omega t + \varphi_i &= \omega t + \varphi_u + {{1}\over{2}}\pi \\ \varphi_i &= \varphi_u + {{1}\over{2}}\pi \\ \varphi_u -\varphi_i &= - {{1}\over{2}}\pi \end{align*}

The phase shift of $- {{1}\over{2}}\pi$ can also be seen in figure 6 and figure 5.

Notice:

In order not to mix up the definitions, for AC signals the fraction of RMS voltage by RMS current is called **(apparent) impedance** Z (in German: Scheinwiderstand or Impedanz).
The impedance is generally defined as $$Z = {{U}\over{I}}$$

Only for a pure resistor as a two-terminal network, the impedance Z_R is equal to the value of the resistance: $Z_R=R$.

For the pure capacitive as a two-terminal network, the impedance Z_C is $Z_C={{1}\over{\omega \cdot C}}$.

Fig. 5: time course of instantaneous voltage and current on a capacitance

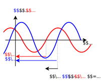


Fig. 6: time course of instantaneous voltage and current on a capacitance

6.3.3 Inductance

The inductance will here be introduced shortly - the detailed introduction is part of electrical engineering 2.
For the capacitance C we had the situation, that it reacts to a voltage change ${{\rm d}\over{{\rm d}t}}u(t)$ with a counteracting current: $$i(t)= C \cdot {{\rm d}\over{{\rm d}t}}u(t)$$ This is due to the fact, that the capacity stores charge carriers q. It appears that "the capacitance does not like voltage changes and reacts with a compensating current". When the voltage on a capacity drops, the capacity supplies a current - when the voltage rises the capacity drains a current.

For an inductance L it is just the other way around: "the inductance does not like current changes and reacts with a compensating voltage drop". Once the current changes the inductance will create a voltage drop that counteracts and continues the current: A current change ${{\rm d}\over{{\rm d}t}}i(t)$ leads to a voltage drop $u(t)$: $$u(t)= L \cdot {{\rm d}\over{{\rm d}t}}i(t)$$ The proportionality factor here is L, the value of the inductance, and it is measured in $[L] = 1~\rm H = 1~Henry$.

We can now again insert the functions representing the instantaneous signals and calculate the derivative: \begin{align*} \sqrt{2}{U}\cdot \sin(\omega t + \varphi_u) &= L \cdot {{\rm d}\over{{\rm d}t}}\left(\sqrt{2}{I}\cdot \sin(\omega t + \varphi_i) \right) \\ &= L \cdot \sqrt{2}{I}\cdot \omega \cdot \cos(\omega t + \varphi_i) \\ \\ {U}\cdot \sin(\omega t + \varphi_u) &= L \cdot {I}\cdot \omega \cdot \sin(\omega t + \varphi_i + {{1}\over{2}}\pi) \tag{6.3.2} \end{align*}

Equating coefficients in $(6.3.2)$ leads to: \begin{align*} U &= L \cdot {I}\cdot \omega \\ \boxed {Z_L = {{U}\over{I}} = \omega \cdot L} \end{align*} and: \begin{align*} \omega t + \varphi_u &= \omega t + \varphi_i + {{1}\over{2}}\pi \\ \varphi_u &= \varphi_i + {{1}\over{2}}\pi \\ \boxed{\varphi = \varphi_u -\varphi_i = + {{1}\over{2}}\pi } \end{align*}

The phase shift of $+ {{1}\over{2}}\pi$ can also be seen in figure 8 and figure 7.

Fig. 7: time course of instantaneous voltage and current on an inductance

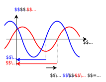


Fig. 8: time course of instantaneous voltage and current on an inductance

Notice:

Remember the formulas for the different pure loads:

Load		**impedance $Z={{U}\over{I}}$**	**phase φ**
Resistance	R	$Z_R = R $	$\varphi_R = 0$
Capacitance	C	$Z_C = {{1}\over{\omega \cdot C}}$	$\varphi_C = - {{1}\over{2}}\pi $
Inductance	L	$Z_L = \omega \cdot L $	$\varphi_L = + {{1}\over{2}}\pi $

Tab. 1: Formulas for the different pure loads

One way to memorize the phase shift is given by the word **CIVIL**:

- **CIVIL**: for a capacitance C the current I leads the voltage V.

Therefore the phase angle φ_I of the current is larger than the phase angle φ_U of the voltage: $\rightarrow \varphi = \varphi_U - \varphi_I < 0$.

- **CIVIL**: for an inductance L the voltage V leads the current I.
  Therefore the phase angle φ_U of the voltge is larger than the phase angle φ_I of the current: $\rightarrow \varphi = \varphi_U - \varphi_I > 0$.

For the concept of AC two-terminal networks, we are also able to use the DC methods of network analysis to solve AC networks.

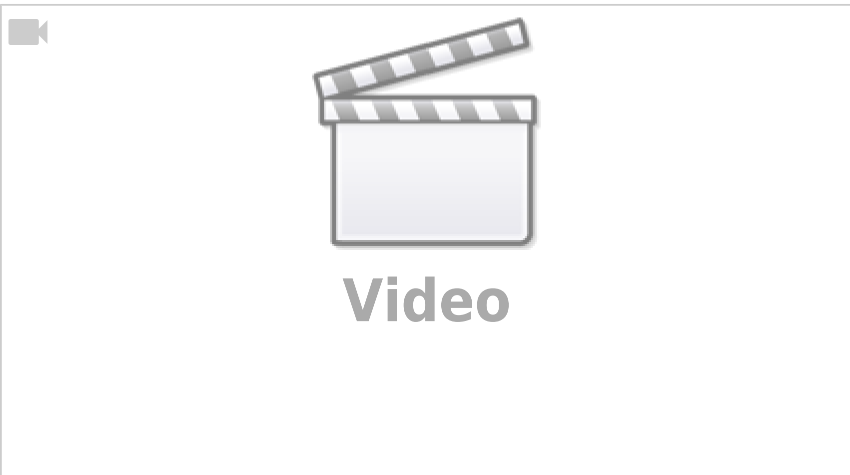
6.4 Complex Values in Electrical Engineering

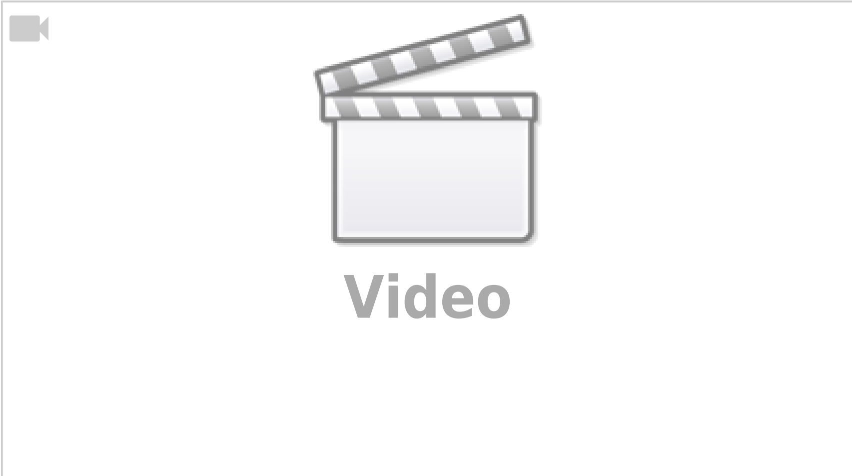
Learning Objectives

By the end of this section, you will be able to:

1. know how sine variables can be symbolized by a vector.
2. know which parameters can determine a sinusoidal quantity.
3. graphically derive a pointer diagram for several existing sine variables.
4. plot the phase shift on the vector and time plots.
5. add sinusoidal quantities in vector and time representation.
6. know and apply the impedance of components.
7. know the frequency dependence of the impedance of the components. In particular, you should know the effect of the ideal components at very high and very low frequencies and be able to apply it for plausibility checks.

The following two videos explain the basic terms of the complex AC calculus: Impedance, Reactance, Resistance





6.4.1 Representation and Interpretation

Up to now, we used for the AC signals the formula $x(t)= \sqrt{2} X \cdot \sin (\omega t + \varphi_x)$ - which was quite obvious.

However, there is an alternative way to look at the alternating sinusoidal signals. For this, we look first at a different, but already a familiar problem (see figure 9).

1. A mechanical, linear spring with the characteristic constant D is displaced due to a mass m in the Earth's gravitational field. The deflection only based on the gravitational field is X_0.
2. At the time $t_0=0$, we deflect this spring a bit more to $X_0 + x(t_0)=X_0 + \hat{X}$ and therefore induce energy into the system.
3. When the mass is released, the mass will spring up and down for $t>0$. The signal can be shown as a shadow when the mass is illuminated sideways.
   For $t>0$, the energy is continuously shifted between potential energy (deflection $x(t)$ around X_0) and kinetic energy (${{\rm d}\over{{\rm d}t}}x(t)$)
4. When looking onto the course of time of $x(t)$, the signal will behave as: $x(t)= \hat{X} \cdot \sin (\omega t + \varphi_x)$
5. The movement of the shadow can also be created by the sideways shadow of a stick on a rotating disc.
   This means, that a two-dimensional rotation is reduced down to a single dimension.

Fig. 9: interpretation of sinusoidal deflection of a spring

The transformation of the two-dimensional rotation to a one-dimensional sinusoidal signal is also shown in figure 10.

Fig. 10: Creation of the sinusoidal signal from a rotational movement

Click on the box “animate?”
press here for the animation

The two-dimensional rotation can be represented with a complex number in Euler's formula. It combines the exponential representation with real part $\Re$ and imaginary part $\Im$ of a complex value: $$ \underline{x}(t)=\hat{X}\cdot {\rm e}^{{\rm j}(\omega t + \varphi_x)} = \Re(\underline{x}) + {\rm j}\cdot \Im(\underline{x})$$

For the imaginary unit ${\rm i}$ the letter ${\rm j}$ is used in electrical engineering since the letter ${\rm i}$ is already taken for currents.

Fig. 11: representation of a phasor on the complex plane

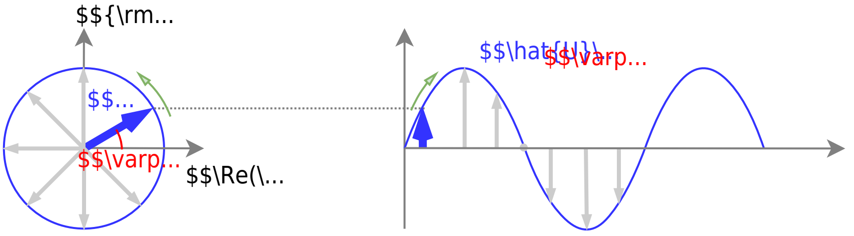
6.4.2 Complex Current and Voltage

The concepts of complex numbers shall now be applied to voltages and currents. Up to now, we used the following formula to represent alternating voltages:

$$u(t)= \sqrt{2} U \cdot \sin (\varphi)$$

This is now interpreted as the instantaneous value of a complex vector $\underline{u}(t)$, which rotates given by the time-dependent angle $\varphi = \omega t + \varphi_u$.

Fig. 12: representation of a voltage phasor on the complex plane



The parts on the complex plane are then given by:

1. The real part $\Re{(\underline{u}(t))} = \sqrt{2}U \cdot \cos (\omega t + \varphi_u)$
2. The imaginary part $\Im{(\underline{u}(t))} = \sqrt{2}U \cdot \sin (\omega t + \varphi_u)$

This is equivalent to the complex phasor $\underline{u}(t)=\sqrt{2}U \cdot {\rm e} ^{{\rm j} (\omega t + \varphi_u)}$

The complex phasor can be separated: \begin{align*} \underline{u}(t) &=\sqrt{2} U \cdot {\rm e}^{{\rm j} (\omega t + \varphi_u)} \\ &=\sqrt{2}\color{blue} {U \cdot {\rm e}^{{\rm j} \varphi_u}} \cdot {\rm e}^{{\rm j} \omega t} \\ &=\sqrt{2}\color{blue}{\underline{U}} \cdot {\rm e}^{{\rm j} \omega t} \\ \end{align*}

The **fixed phasor** (in German: *komplexer Festzeiger*) of the voltage is given by $\color{blue}{\underline{U}}= \color{blue}{U \cdot e ^{j \varphi_u}} $

Generally, from now on not only the voltage will be considered as a phasor, but also the current $\underline{I}$ and derived quantities like the impedance $\underline{X}$.
Therefore, the known properties of complex numbers from Mathematics 101 can be applied:

- A multiplication with j equals a phase shift of $+90°$
- A multiplication with ${{1}\over{j}}$ equals a phase shift of $-90°$

6.5 Complex Impedance

> **Learning Objectives**
>
> By the end of this section, you will be able to:
>
> 1. draw and read pointer diagrams.
> 2. know and apply the complex value formulas of impedance, reactance, and resistance.

6.5.1 Introduction to Complex Impedance

The complex impedance is "nearly" similar calculated like the resistance. In the subchapters before, that impedance Z was calculated by $Z=\frac{U}{I}$.
Now the complex impedance is:

\begin{align*} \underline{Z}&=\frac{\underline{U}}{\underline{I}} \\ &= \Re{(\underline{Z})} + {\rm j} \cdot \Im{(\underline{Z})} \\ &= R + {\rm j} \cdot X \\ &= Z \cdot {\rm e}^{{\rm j} \varphi} \\ &= Z \cdot (\cos \varphi + j \cdot \sin \varphi) \end{align*}

With

- the resistance R (in German: *Widerstand*) as the pure real part
- the reactance X (in German: *Blindwiderstand*) as the pure imaginary part
- the impedance Z (in German: *Scheinwiderstand*) as the complex number given by the <u>complex</u> addition of resistance and the reactance as a complex number

The impedance can be transformed from Cartesian to polar coordinates by:

- $Z=\sqrt{R^2 + X^2}$
- $\varphi = \arctan \frac{X}{R} $

The other way around it is possible to transform by:

- $R = Z \cos \varphi$
- $X = Z \sin \varphi$

value - and therefore a phasor - can simply

6.5.2 Application on pure Loads

With the complex impedance in mind, the table 1 can be expanded to:

Load 		**integral representation **	**complex impedance $\underline{Z}={{\underline{U}}\over{\underline{I}}}$**	**impedance $Z $**	**phase $\varphi $**
Resistance	R	${u} = R \cdot {i}$	$Z_R = R $	$Z_R = R $	$\varphi_R = 0$
Capacitance	C	${u} ={{1}\over{C}}\cdot \int {\rm i} dt$	$Z_C = {{1}\over{{\rm j}\omega \cdot C}} = {{-{\rm j}}\over{\omega \cdot C}}$	$Z_C = {{1}\over{\omega \cdot C}}$	$\varphi_C = -{{1}\over{2}}\pi \hat{=} -90°$
Inductance	L	${u} = L \cdot {{\rm d}\over{{\rm d}t}} {i}$	$Z_L = {\rm j} \omega \cdot L $	$Z_L = \omega \cdot L $	$\varphi_L = +{{1}\over{2}}\pi \hat{=} +90°$

Tab. 2: Formulas for the different pure loads

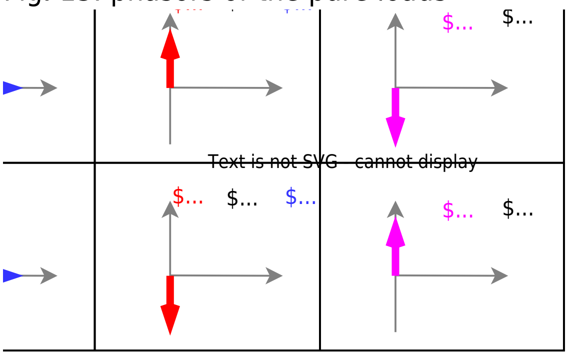
The relationship between ${\rm j}$ and integral calculus should be clear:

1. The derivative of a sinusoidal value - and therefore a phasor - can simply be written as “$\cdot {\rm j}\omega$”,
   which also means a phase shift of $+90°$:
   \begin{align*}{{\rm d}\over{{\rm d}t}} {\rm e}^{{\rm j}(\omega t + \varphi_x)} = {\rm j} \cdot {\rm e}^{{\rm j}(\omega t + \varphi_x)}\end{align*}
2. The integral of a sinusoidal value - and therefore a phasor - can simply be written as “$\cdot (-{{1}\over{ {\rm j}\omega}})$”,
   which also means a phase shift of $-90°$.[1]

   \begin{align*} \int {\rm e}^{{\rm j}(\omega t + \varphi_x)} = {{1}\over{\rm j\omega}} \cdot {\rm e}^{{\rm j}(\omega t + \varphi_x)} = -{{\rm j}\over{\omega}} \cdot {\rm e}^{{\rm j}(\omega t + \varphi_x)} \end{align*}

Once a fixed input voltage is given, the voltage phasor $\underline{U}$, the current phasor $\underline{I}$, and the impedance phasor $\underline{Z}$. In figure 13 these phasors are shown.

Fig. 13: phasors of the pure loads

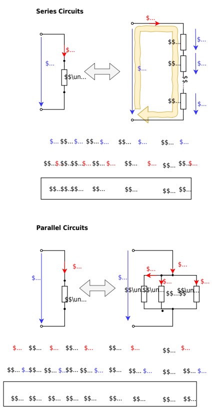


6.5.3 Application on Impedance Networks

Simple Networks

In the chapter Kirchhoff's Circuit Laws we already had a look at simple networks like a series or parallel circuit of resistors.
These formulas not only apply to ohmic resistors but also to impedances:

Fig. 14: Simple Networks

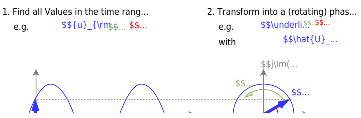


Similarly, the voltage divider, the current divider, the star-delta transformation, and the Thevenin and Northon Theorem can be used, by substituting resistances with impedances. This means for example, every linear source can be represented by an output impedance $\underline{Z}_o$ and an ideal voltage source $\underline{U}$.

More "complex" Networks

For more complex problems having AC values in circuitries, the following approach is beneficial. This concept will be used in the next chapter and in circuit design.

Fig. 15: Approach for AC cicruitries



Notice:

For a complex number are always two values are needed. These are either

1. the real part (e.g. the resistance) and the imaginary part (e.g. the reactance), or
2. the absolute value (e.g. the absolute value of the impedance) and the phase

Therefore, instead of the form $\underline{Z}=Z\cdot {\rm e}^{{\rm j}\varphi}$ for the phasors often the form $Z\angle{\varphi}$ is used.

Exercises

Exercise 6.3.1 Impedance of single Components I

A coil has a reactance of 80Ω at a frequency of $500 ~\rm Hz$. At which frequencies the impedance will have the following values?

1. $85 ~\Omega$
2. $120 ~\Omega$
3. $44 ~\Omega$

Solution

When the frequency changes the reactance changes but the inductance is constant.
Therefore, the inductance is needed.
It can be calculated by the given reactance for $f_0 = 500 ~\rm Hz$. \begin{align*}

X_{L0}&=2\pi f_0L\\ L &=\frac{X_{L0}}{2\pi f_0} \end{align*}

On the other hand, one can also use the rule of proportion here, and circumvent the calculation of inductance.
It is possible to calculate the reactance at other frequencies with the given reactance.
\begin{align*} X_L&=2\pi fL\\ f &=\frac{X_L}{2\pi L}\\ &=\frac{X_L}{X_{L0}}f_0
\end{align*}

With the values given: \begin{equation*} f_1 = \frac{85
~\Omega}{80~\Omega}\cdot500~{\rm Hz}\qquad f_2 =
\frac{120~\Omega}{80~\Omega}\cdot500~{\rm Hz}\qquad f_3 = \frac{44
~\Omega}{80~\Omega}\cdot500~{\rm Hz} \end{equation*}

Final value

\begin{equation*} f_1=531.25~{\rm Hz}\qquad f_2=750~{\rm Hz}\qquad f_3=275~{\rm Hz} \end{equation*}

Exercise 6.3.2 Impedance of single Components II

A capacitor with $5 ~{\rm µF}$ is connected to a voltage source which generates $U_\sim = 200 ~{\rm V}$. At which frequencies the following currencies can be measured?

1. $0.5 ~\rm A$
2. $0.8 ~\rm A$
3. $1.3 ~\rm A$

Exercise 6.3.3 Impedance of single Components III

A capacitor shall have a capacity of $4.7 ~{\rm µF} \pm 10~\%$. This capacitor shall be used with an AC voltage of $400~\rm V$ and $50~\rm Hz$. What is the possible current range which could be found on this component?

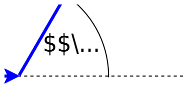
Exercise 6.5.1 Two voltage sources

Two ideal AC voltage sources 1 and 2 shall generate the RMS voltage drops $U_1 = 100~\rm V$ and $U_2 = 120~\rm V$.
The phase shift between the two sources shall be $+60°$. The phase of source 1 shall be $\varphi_1=0°$.
The two sources shall be located in series.

1. Draw the phasor diagram for the two voltage phasors and the resulting phasor.

Solution 1

The phasor diagram looks roughly like this:



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2. Calculate the resulting voltage and phase.

Solution 2

By the law of cosine, we get: \begin{align*} U&= \sqrt{{{U_1 }^2}+{{U_1 }^2}-2{U_1 } \cdot{U_2 }\cdot \cos(180°- {\varphi_2})} \\ &= \sqrt{{{100~V}^2}+{{120~V}^2}-2\cdot{100~V} \cdot{120~V}\cdot \cos(120°)} \end{align*}

The angle is by the tangent of the relation of the imaginary part to the real part of the resulting voltage. \begin{align*} \varphi&= \arctan 2 \left(\frac{Im\{\underline{U}\}}{Re\{\underline{U}\}} \right)\\ &= \arctan 2 \left(\frac{Im\{\underline{U}_1\}+Im\{\underline{U}_2\}}{Re\{\underline{U}_1\}+\{\underline{U}_2\}} \right)\\ &= \arctan 2 \left(\frac{{U}_2 \cdot\sin(\varphi_2)}{{U}_1 +{U}_2\cos(\varphi_2)} \right)\\ &= \arctan 2 \left(\frac{120~{\rm V}\cdot\sin(60°)}{100~{\rm V}+120~V\cos(60°)} \right) \end{align*}

Final value

\begin{align*} U &=190.79~{\rm V} \\ \varphi&=33° \end{align*}

3. Is the resulting voltage the RMS value or the amplitude?

Solution 3

The resulting voltage is the RMS value.

The source 2 shall now be turned around (the previous plus pole is now the minus pole and vice versa).

4. Draw the phasor diagram for the two voltage phasors and the resulting phasor for the new circuit.

Solution 4

The phasor diagram looks roughly like this.
But have a look at the solution for question 5!

nderline{U_2}$$

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5. Calculate the resulting voltage and phase.

Solution 5

By the law of cosine, we get: $$\begin{align*} U&= \sqrt{{{U_1 }^2}+{{U_1 }^2}-2{U_1 } \cdot{U_2 }\cdot \cos(180°- {\varphi_1})} &= \sqrt{{{100~V}^2}+{{120~V}^2}-2\cdot{100~V} \cdot{120~V}\cdot \cos(60°)} \end{align*}$$ The angle is by the tangent of the relation of the imaginary part to the real part of the resulting voltage. $$\begin{align*} \varphi&= \arctan 2 \left(\frac{Im\{\underline{U}\}} {Re\{\underline{U}\}} \right)\\ &= \arctan 2 \left(\frac{Im\{\underline{U}_1\}+Im\{\underline{U}_2\}}{Re\{\underline{U}_1\}+\{\underline{U}_2\}} \right)\\ &= \arctan 2 \left(\frac{-{U}_2 \cdot\sin(\varphi_2)}{{U}_1 -{U}_2\cos(\varphi_2)} \right)\\ &= \arctan 2 \left(\frac{-120~{\rm V}\cdot\sin(60°) }{100~{\rm V}-120~V\cos(60°)} \right) \\ &= \arctan 2 \left(\frac{-103.92...~{\rm V} }{+40~{\rm V}} \right) \end{align*}$$ The calculated (positive) horizontal and (negative) vertical dimension for the voltage indicates a phasor in the fourth quadrant. Does it seem right?
The phasor diagram which was shown in answer 4. cannot be correct.
With the correct lengths and angles, the real phasor diagram looks like this:

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Here the phasor is in the fourth quadrant with a negative angle.

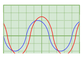
Final value

$$\begin{align*} U &=111.355~{\rm V}\\ \varphi&=-68.948° \end{align*}$$

Notice:
Be aware that some of the calculators only provide $\tan^{-1}$ or $\arctan$ and not $\arctan2$!
Therefore, you have always to check whether the solution lies in the correct quadrant.

Exercise 6.5.2 oscilloscope plot

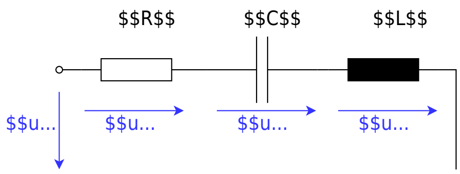
The following plot is visible on an oscilloscope (= plot tool for voltages and current).



1. What is the RMS value of the current and the voltage? What is the frequency f and the phase φ? Does the component under test behave ohmic, capacitive, or inductive?
2. How would the equivalent circuit look like, when it is built by two series components?
3. Calculate the equivalent component values (R, C or L) of the series circuit.
4. How would the equivalent circuit look like, when it is built by two parallel components?
5. Calculate the equivalent component values (R, C or L) of the parallel circuit.

Exercise 6.5.3 Series Circuit

The following circuit shall be given.

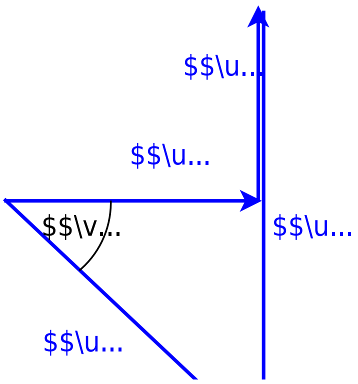


This circuit is used with different component values, which are given in the following. Calculate the RMS value of the missing voltage and the phase shift φ between U and I.

1. $U_R = 10~\rm V$, $U_L = 10~\rm V$, $U_C = 20~\rm V$, $U=\rm ?$

Solution

The drawing of the voltage pointers is as follows:



The voltage U is determined by the law of Pythagoras \begin{align*} U &= \sqrt{{{U_R } ^2}+{({U_L }-{U_C }})^2} \\ &= \sqrt{(10~{\rm V})^2+ {({10~{\rm V}}-{20~{\rm V}}})^2} \end{align*} The phase shift angle is calculated by simple geometry. \begin{align*} \tan(\varphi)&=\frac{{U_L }-{U_C }}{U_R}\\ &=\frac{{10~{\rm V}}-{20~{\rm V}}}{10~{\rm V}} \end{align*} Considering that the angle is in the fourth quadrant we get:

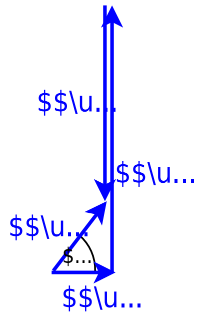
Final value

\begin{equation*} U=\sqrt{2}\cdot 10~{\rm V} = 14.1~{\rm V} \qquad \varphi=-45° \end{equation*}

2. $U_R = ?$, $U_L = 150~\rm V$, $U_C = 110~\rm V$, $U=50~\rm V$

Solution 2

The drawing of the voltage pointers is as follows:

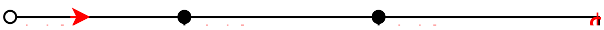


The voltage U_R is determined by the law of Pythagoras \begin{align*} U_R&=\sqrt{{U ^2}+{({U_L} -{U_C}})^2}\\ &=\sqrt{(50~\rm V)^2 +{(150~\rm V -110~\rm V})^2} \end{align*} The phase shift angle is calculated by simple geometry. \begin{align*} \tan(\varphi)&=\frac{{U_L} -{U_C} }{U_R}\\ &=\frac{{150~\rm V}-{110~\rm V}}{30~\rm V} \end{align*} Considering that the angle is in the fourth quadrant we get:

Final value

\begin{equation*} U_R= 30~{\rm V}\qquad \varphi=53.13° \end{equation*}

Exercise 6.5.4 Parallel Circuit

The following circuit shall be given. 

in the following, some of the numbers are given. Calculate the RMS value of the missing currents and the phase shift φ between U and I.

1. $I_R = 3~\rm A$, $I_L = 1 ~\rm A$, $I_C = 5 ~\rm A$, $I=?$
2. $I_R = ?$, $I_L = 1.2~\rm A$, $I_C = 0.4~\rm A$, $I=1~\rm A$

Exercise 6.5.5 Complex Calculation I

The following two currents with similar frequencies, but different phases have to be added. Use complex calculation!

- $i_1(t) = \sqrt{2} \cdot 2 ~A \cdot \cos (\omega t + 20°)$
- $i_2(t) = \sqrt{2} \cdot 5 ~A \cdot \cos (\omega t + 110°)$

Exercise 6.5.6 Complex Calculation II

Two complex impedances $\underline{Z}_1$ and $\underline{Z}_2$ are investigated. The resulting impedance for a series circuit is $60~\Omega + \rm j \cdot 0 ~\Omega $. The resulting impedance for a parallel circuit is $25~\Omega + \rm j \cdot 0 ~\Omega $.

Result

What are the values for $\underline{Z}_1$ and $\underline{Z}_2$?

\begin{align*} R_1 &= 30~\Omega \\ R_2 &= 30~\Omega \\ X_1 &= \mp \sqrt{600}~\Omega \approx \mp 24.5~\Omega \\ X_2 &= \pm \sqrt{600}~\Omega \approx \pm 24.5~\Omega \\ \end{align*}

Solution

It's a good start to write down all definitions of the given values:

- the given values for the series circuit ($\square_\rm s$) and the parallel circuit ($\square_\rm p$) are: \begin{align*} R_\rm s = 60 ~\Omega , \quad X_\rm s = 0 ~\Omega \\ R_\rm p = 25 ~\Omega , \quad X_\rm p = 0 ~\Omega \\ \end{align*}
- the series circuit and the parallel circuit results into: \begin{align*} R_{\rm s} = \underline{Z}_1 + \underline{Z}_2 \tag{1} \\ R_{\rm p} = \underline{Z}_1 || \underline{Z}_2 \tag{2} \\ \end{align*}
- the unknown values of the two impedances are: \begin{align*} \underline{Z}_1 = R_1 + {\rm j}\cdot X_1 \tag{3} \\ \underline{Z}_2 = R_2 + {\rm j}\cdot X_2 \tag{4} \\ \end{align*}

Based on (1),(3) and (4): \begin{align*} R_\rm s &= \underline{Z}_1 &&+ \underline{Z}_2 \\ &= R_1 + {\rm j}\cdot X_1 &&+ R_2 + {\rm j}\cdot X_2 \\ \rightarrow 0 &= R_1 + R_2 - R_\rm s &&+ {\rm j}\cdot (X_1 + X_2) \\ \end{align*} Real value and imaginary value must be zero: \begin{align*} R_1 &= R_{\rm s} - R_2 \tag{5} \\ X_1 &= - X_2 \tag{6} \end{align*}

Based on (2) with $R_\rm s = \underline{Z}_1 + \underline{Z}_2$ (1): \begin{align*} R_{\rm p} &= {{\underline{Z}_1 \cdot \underline{Z}_2}\over{\underline{Z}_1 + \underline{Z}_2}} \\ &= {{\underline{Z}_1 \cdot \underline{Z}_2}\over{R_\rm s}} \\ \\ R_{\rm p} \cdot R_{\rm s} &= \underline{Z}_1 \cdot \underline{Z}_2 \\ &= (R_1 + {\rm j}\cdot X_1)\cdot (R_2 + {\rm j}\cdot X_2) \\ &= R_1 R_2 + {\rm j}\cdot (R_1 X_2 + R_2 X_1) - X_1 X_2 \\ \end{align*}

Substituting R_1 and X_1 based on (5) and (6): \begin{align*} R_{\rm p} \cdot R_{\rm s} &= (R_{\rm s} - R_2) R_2 + {\rm j}\cdot ((R_{\rm s} - R_2) X_2 - R_2 X_2) + X_2 X_2 \\ \rightarrow 0 &= R_{\rm s} R_2 - R_2^2 + X_2^2 - R_{\rm p} \cdot R_{\rm s} + {\rm j}\cdot ((R_{\rm s} - R_2) X_2 - R_2 X_2) \\ \end{align*}

Again real value and imaginary value must be zero: \begin{align*} 0 &= j\cdot ((R_{\rm s} - R_2) X_2 - R_2 X_2) \\ &= R_{\rm s}X_2 - 2 \cdot R_2 X_2 \\ \rightarrow R_2 = {{1}\over{2}} R_{\rm s} \tag{7} \\ \\ 0 &= R_{\rm s} R_2 - R_2^2 + X_2^2 - R_{\rm p} \cdot R_{\rm s} \\ &= R_{\rm s} ({{1}\over{2}} R_{\rm s}) - ({{1}\over{2}} R_{\rm s})^2 - X_2^2 - R_{\rm p} \cdot R_{\rm s} \\ &= {{1}\over{4}} R_{\rm s}^2 + X_2^2 - R_{\rm p} \cdot R_{\rm s} \\ \rightarrow X_2 = \pm \sqrt{R_{\rm p} \cdot R_{\rm s} - {{1}\over{4}} R_{\rm s}^2 } \tag{8} \\ \\ \end{align*}

The concluding result is: \begin{align*} (5)+(7): \quad R_1 &= {{1}\over{2}} R_{\rm s} \\ (7): \quad R_2 &= {{1}\over{2}} R_{\rm s} \\ (6)+(8) \quad X_1 &= \mp \sqrt{R_{\rm p} \cdot R_{\rm s} - {{1}\over{4}} R_{\rm s}^2 } \\ (8): \quad X_2 &= \pm \sqrt{R_{\rm p} \cdot R_{\rm s} - {{1}\over{4}} R_{\rm s}^2 } \end{align*}

Exercise 6.5.7 real Coils I

A real coil has both ohmic and inductance behavior. At DC voltage the resistance is measured as $9 ~\Omega$. With an AC voltage of $5~\rm V$ at $50~\rm Hz$ a current of $0.5~\rm A$ is measured.

What is the value of the inductance L?

Exercise 6.5.8 real Coils II

A real coil has both ohmic and inductance behavior. This coil has at $100~\rm Hz$ an impedance of $1.5~\rm k\Omega$ and a resistance $1~\rm k\Omega$.

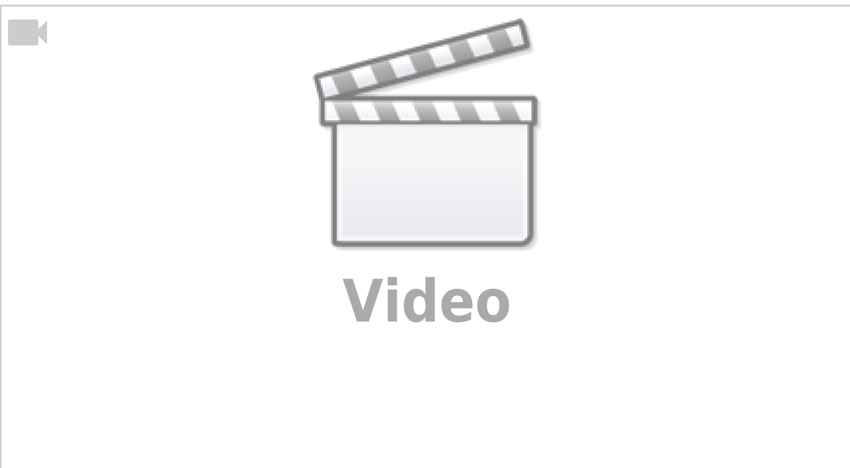
What is the value of the reactance and inductance?

Exercise 6.5.9 Capacitors and Resistance I

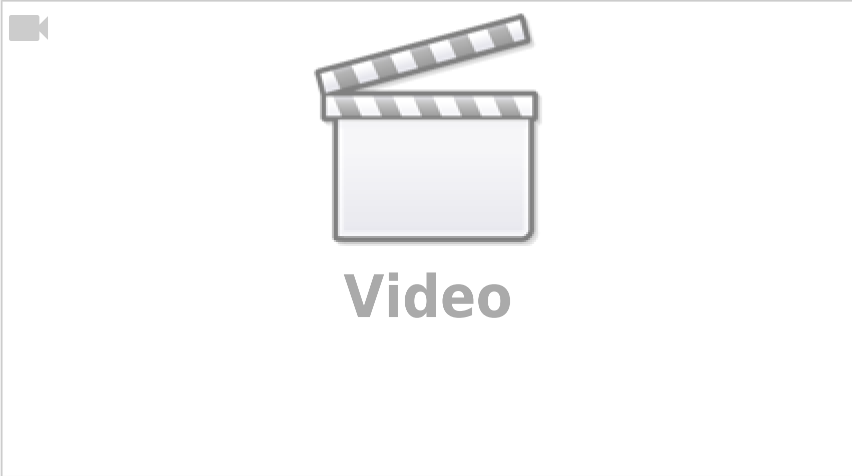
An ideal capacitor is in series with a resistor $R=1~\rm k\Omega$. The capacitor shows a similar voltage drop to the resistor for $100~\rm Hz$.

What is the value of the capacitance?

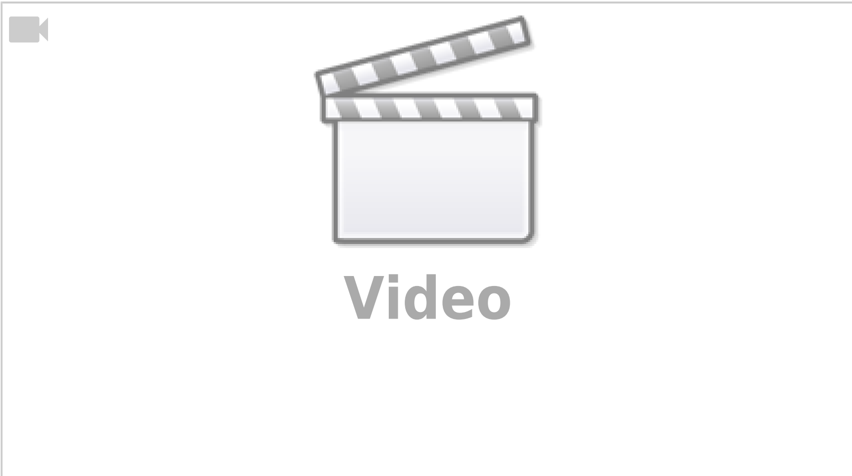
Exercise 6.6.1 Impedance in Series Circuit of multiple Components I



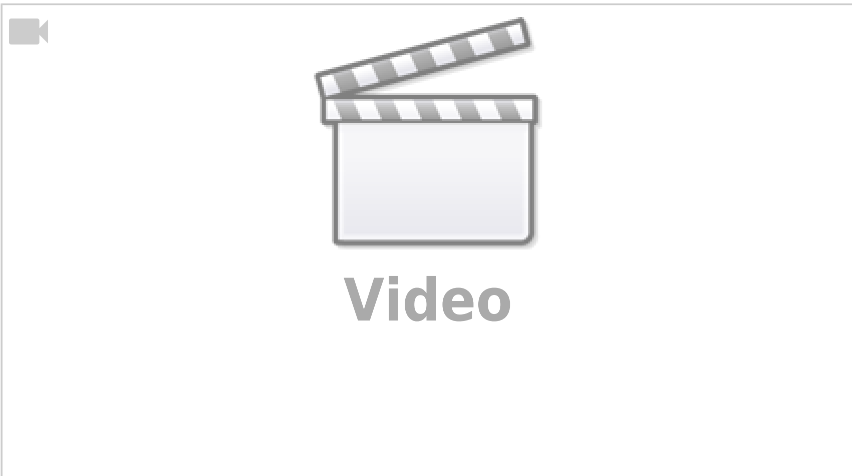
Exercise 6.6.2 Impedance in Series Circuit of multiple Components II



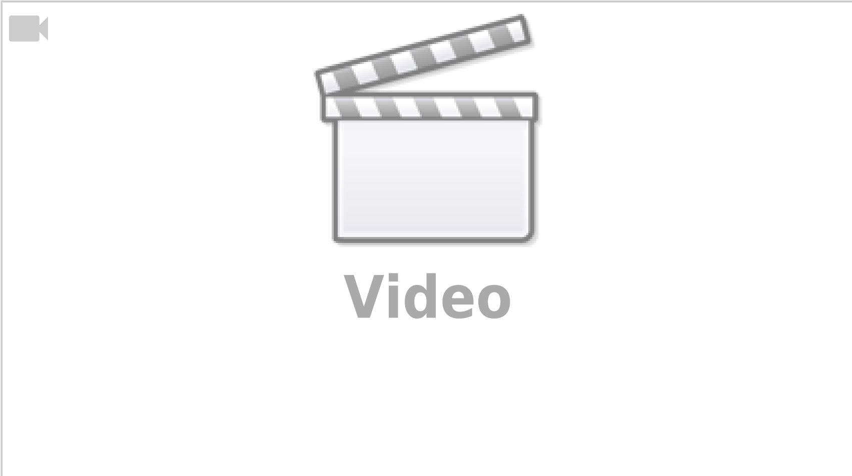
Exercise 6.6.3 Impedance in Parallel Circuit of multiple Components I



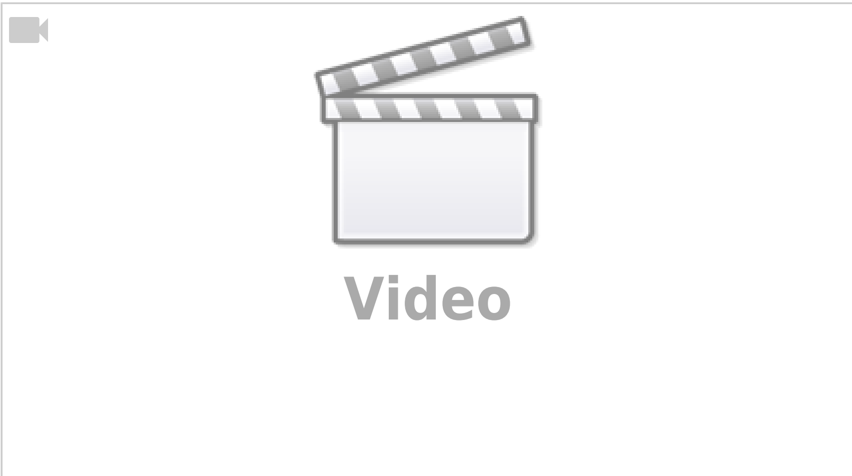
Exercise 6.6.4 Impedance in Mixed Parallel and Series Circuit of multiple Components I



Exercise 6.6.5 Impedance in Mixed Parallel and Series Circuit of multiple Components II



Exercise 6.6.6 Impedance in Mixed Parallel and Series Circuit of multiple Components III



1)
in general, here the integration constant must be considered. This is however often neglectable since only AC values (without a DC value) are considered.

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