6 Inductances in Circuits

Student Group

First Name	Surname	Matrikel Nr.

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6.1 Basic Circuits

Focus here: uncoupled inductors!

6.1.1 Series Circuits

Based on $L = {{ \Psi(t)}\over{i}}$ and Kirchhoff's mesh law ($i=\rm const$) the series circuit of inductions can be interpreted as a single current i which generates multiple linked fluxes Ψ. Since the current must stay constant in the series circuit, the following applies for the equivalent inductor of a series connection of single ones:

$$\begin{align*} L_{\rm eq} &= {{\sum_i \Psi_i}\over{I}} = \sum_i L_i \end{align*}$$

A similar result can be derived from the induced voltage $u_{ind}= L {{{\rm d}i}\over{{\rm d}t}}$, when taking the situation of a series circuit (i.e. $i_1 = i_2 = i_1 = ... = i_{\rm eq}$ and $u_{\rm eq}= u_1 + u_2 + ...$):

$$\begin{align*} & u_{\rm eq} & = &u_1 & + &u_2 &+ ... \\ & L_{\rm eq} {{{\rm d}i_{\rm eq} }\over{{\rm d}t}} & = &L_{1} {{{\rm d}i_{1} }\over{{\rm d}t}} & + &L_{2} {{di_{2} }\over{dt}} &+ ... \\ & L_{\rm eq} {{{\rm d}i }\over{{\rm d}t}} & = &L_{1} {{{\rm d}i }\over{{\rm d}t}} & + &L_{2} {{di }\over{dt}} &+ ... \\ & L_{\rm eq} & = &L_{1} & + &L_{2} &+ ... \\ \end{align*}$$

6.1.2 Parallel Circuits

For parallel circuits, one can also start with the principles based on Kirchhoff's mesh law:

$$\begin{align*} u_{\rm eq}= u_1 = u_2 = ... \\ \end{align*}$$

and Kirchhoff's nodal law:

$$\begin{align*} i_{\rm eq}= i_1 + i_2 + ... \\ \end{align*}$$

Here, the formula for the induced voltage has to be rearranged:

$$\begin{align*} u_{\rm ind} &= L {{{\rm d}i}\over{{\rm d}t}} \quad \quad \quad \quad \bigg| \int(){\rm d}t \\ \int u_{\rm ind} {\rm d}t &= L \cdot i \\ i &= {{1}\over{L}} \cdot \int u_{\rm ind} {\rm d}t \\ \end{align*}$$

By this, we get:

$$\begin{align*} i_{\rm eq} &=& i_1 &+& i_2 &+& ... \\ {{1}\over{L_{\rm eq}}} \cdot \int u_{\rm eq} {\rm d}t &=& {{1}\over{L_1}} \cdot \int u_{1} {\rm d}t &+& {{1}\over{L_2}} \cdot \int u_{2} {\rm d}t &+& ... \\ {{1}\over{L_{\rm eq}}} \cdot \int u {\rm d}t &=& {{1}\over{L_1}} \cdot \int u {\rm d}t &+& {{1}\over{L_2}} \cdot \int u {\rm d}t &+& ... \\ {{1}\over{L_{\rm eq}}} &=& {{1}\over{L_1}} &+& {{1}\over{L_2}} &+& ... \\ \end{align*}$$

Notice:

The inductor behaves in the parallel and series circuit similar to the resistor.

6.1.3 in AC Circuits

For AC circuits (i.e. with sinusoidal signals) the impedance Z based on the real part R and imaginary part X has to be considered. To do so, one has to solve:

\begin{align*} \underline{Z} = {{\underline{u}}\over{\underline{i}}} = {{1}\over{\underline{i}}} \cdot \underline{u} \end{align*}

With the induction $u_{\rm ind}= L {{{\rm d}i}\over{{\rm d}t}}$ we get:

\begin{align*} \underline{Z} &= {{1} \over {\underline{i}}} \cdot L { { {\rm d}\underline{i} } \over {{\rm d}t} } \\ \end{align*}

Without limiting the generality, one can assume the current i to be: $i = I \cdot \sqrt{2} \cdot {\rm e}^{{\rm j} \cdot \omega t + \varphi_0}$.
Once inserted, the formula gets:

\begin{align*} \underline{Z} &= {{1} \over {I \cdot \sqrt{2} \cdot {\rm e}^{{\rm j} \cdot \omega t + \varphi_0}}} \cdot L {{ {\rm d}} \over {{\rm d}t} } \left(I \cdot \sqrt{2} \cdot {\rm e}^{{\rm j} \cdot \omega t + \varphi_0} \right) \\ &= {{1} \over {I \cdot \sqrt{2} \cdot {\rm e}^{{\rm j} \cdot \omega t + \varphi_0}}} \cdot L I \cdot \sqrt{2} \cdot {{ {\rm d}} \over {{\rm d}t} } \left({\rm e}^{{\rm j} \cdot \omega t + \varphi_0} \right) \\ &= {{1} \over {\qquad\quad\; {\rm e}^{{\rm j} \cdot \omega t + \varphi_0}}} \cdot L \qquad\; \cdot {{ {\rm d}} \over {{\rm d}t} } \left({\rm e}^{{\rm j} \cdot \omega t + \varphi_0} \right) \\ &= {{1} \over {\qquad\quad\; {\rm e}^{{\rm j} \cdot \omega t + \varphi_0}}} \cdot L \qquad\; \cdot j\omega \cdot {\rm e}^{{\rm j} \cdot \omega t + \varphi_0} \\ \\ \underline{Z} &= L \cdot {\rm j}\omega \\ \end{align*}

Notice:
In AC calculations the inductor is described with a complex impedance $\boxed{\underline{Z} = L \cdot j \omega}$

6.2 Charging and Discharging

Charging and discharging an RL circuit is comparable to the RC-circuit in chapter DC Circuit Transients from last semester. Details are not covered here; see OpenStax

6.3 Resonance Phenomena

Similar to last semester's approach, we now focus on circuits with inductors L. For preparation, please recap the chapter Circuits under different Frequencies from last semester.

6.3.1 RLC - Series Resonant Circuit

As seen last semester, the circuits with complex impedances can be interpreted as four-terminal networks. There, we will again look at "output versus input", i.e: $A_V = {{U_O}\over{U_I}} \rightarrow \underline{A}_V = {{\underline{U}_O}\over{\underline{U}_I}} $.

In this chapter, we look at a combination where all three components resistor R, capacitor C, and inductance L are used.

Fig. 1: circuit of the series resonant circuit

If a resistor R, a capacitor C, and an inductance L are connected in series, the result is a **series resonant circuit**. In this case, it is not clearly defined, what the output voltage is. Consequently, it must be considered how the voltages behave across all the individual components in the following. The total voltage (= input voltage U_I) results to:

$$\begin{align*} \underline{U}_I = \underline{U}_R + \underline{U}_L + \underline{U}_C \end{align*}$$

Since the current in the circuit must be constant, the total impedance can be determined here in a simple way:

$$\begin{align*} \underline{U}_I &= R \cdot \underline{I} + {\rm j} \omega L \cdot \underline{I} + \frac {1}{j\omega C } \cdot \underline{I} \\ \underline{U}_I &= \left(R + {\rm j} \omega L - {\rm j} \cdot \frac {1}{ \omega C } \right) \cdot \underline{I} \\ \underline{Z}_{\rm eq} &= R + {\rm j} \omega L - {\rm j} \cdot \frac {1}{ \omega C } \end{align*}$$

By this, the magnitude of the (input) voltage U_I, the (input or total) impedance Z, and the phase result to:

$$\begin{align*} U_I &= \sqrt{U_R^2 + (U_Z)^2} = \sqrt{U_R^2 + (U_L - U_C)^2} \end{align*}$$

$$\begin{align*} Z &= \sqrt{R^2 + X^2} = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} \end{align*}$$

$$\begin{align*} \varphi_u = \varphi_Z &= \arctan \frac{\omega L - \frac{1}{\omega C}}{R} \end{align*}$$

There are now 3 different situations to distinguish:

- If $U_L > U_C$ the whole setup behaves like an ohmic-inductive load. This is the case at high frequencies.
- If $U_L = U_C$, the total input voltage U is applied to the resistor. In this case, the total resistance Z is minimal and only ohmic.
Thus, the current I is then maximal. If the current is maximum, then the responses of the capacitance and inductance - their voltages - are also maximum. This situation is the **resonance case**.
- If $U_L < U_C$ then the whole setup behaves like a resistive-capacitive load. This is the case at low frequencies.

Again, there seems to be a singular frequency, namely when $U_L = U_C$ or $Z_L = Z_C$ holds:

$$\begin{align*} \frac{1}{\omega_0 C} & = \omega_0 L \\ \omega_0 & = \frac{1}{ \sqrt{LC}} \\ 2\pi f_0 & = \frac{1}{ \sqrt{LC}} \rightarrow \boxed{ f_0 = \frac{1}{2\pi \sqrt{LC}} } \end{align*}$$

The frequency f_0 is called **resonance frequency**.

	$\quad$	$f \rightarrow 0$	$\quad$	$f = f_0$	$\quad$	$f \rightarrow \infty$
voltage U_R at the resistor		$\boldsymbol{\small{0}}$		$\boldsymbol{\LARGE{U}}$ since the impedances just cancel out		$ \boldsymbol{\small{0}}$
voltage U_L at the inductor		$\boldsymbol{\small{0}}$ because ωL becomes very small		$\boldsymbol{\omega_0 L \cdot I = \omega_0 L \cdot \frac{U}{R} = \color{blue}{\frac{1}{R}\sqrt{\frac{L}{C}}\cdot U}}$		$\boldsymbol{\LARGE{U}}$ since ωL becomes very large

	$\quad$	$f \rightarrow 0$	$\quad$	$f = f_0$	$\quad$	$f \rightarrow \infty$
voltage U_C at the capacitor		$\boldsymbol{\LARGE{U}}$ because $\frac{1}{\omega C}$ becomes very large		$\boldsymbol{\frac{1}{\omega_0 C} \cdot I = \frac{1}{\omega_0 C} \cdot \frac{U}{R} = \color{blue}{\frac{1}{R}\sqrt{\frac{L}{C}}\cdot U}}$		$\boldsymbol{\small{0}}$ because $\frac{1}{\omega C}$ becomes very small

The calculation in the table shows that in the resonance case, the voltage across the capacitor or inductor deviates from the input voltage by a factor $\color{blue}{\frac{1}{R}\sqrt{\frac{L}{C}}}$. This quantity is called **quality or Q-factor** $Q_{\rm S}$:

\begin{align*} \boxed{ Q_{\rm S} = \left.\frac{U_C}{U} \right\vert_{\omega = \omega_0} = \left.\frac{U_L}{U} \right\vert_{\omega = \omega_0} = \color{blue}{\frac{1}{R}\sqrt{\frac{L}{C}}} } \end{align*}

The quality can be greater than, less than, or equal to 1. The quality $Q_{\rm S}$ does not have a unit and should not be confused with the charge Q.

- If the quality is very high, the overshoot of the voltages at the impedances becomes very large in the resonance case. This is useful and necessary in various applications, e.g. in an RLC element as an antenna.
- If the Q is very small, overshoot is no longer seen. Depending on the impedance at which the output voltage is measured, a high-pass or low-pass is formed similar to the RC or RL element. However, this has a steeper slope in the blocking range. This means that the filter effect is better.

The reciprocal of the Q is called **attenuation** $d_{\rm S}$. This is specified when using the circuit as a non-overshooting filter.

\begin{align*} \boxed{ d_{\rm S} = \frac{1}{Q_{\rm S}} = R \sqrt{\frac{C}{L}} } \end{align*}

Fig. 2: Amplitude and Phase Response of a Series Resonant Circuit

Series Resonant Circuit in Time Domain (Voltage on Inductor)

Series Resonant Circuit in Frequency Domain (Voltage on Inductor)

Series Resonant Circuit in Frequency Domain (Voltage on Capacitor)

Series Resonant Circuit in Frequency Domain (Voltage on Resistor)

Parallel Resonant Circuit in Frequency Domain (Voltage on Resistor)

6.4 Applications of Inductors

- ferrite bead
- Decoupling
- Filter
- unwanted coupling and circuit design

6.5 Examples

Decoupling Capacitor on the Microcontroller

Simulation in Falstad. Note: The simulation gives a highly simplified picture. The response of the microcontroller is shown reduced to a triangular signal, since the slope of the voltages cannot be represented. A real simulation requires a powerful SPICE program in which the conduction theory can be represented.

Further details can be found here (practice), here (layout), also Layout or Layout.

Crystal as Resonant Circuit

more background

Simulation in Time Domain

Simulation in Frequency Domain

setup for the design

Exercises

Exercise 6.3.1 Series Resonant Circuit I

A R-L-C series circuit uses a capacity of $C=100 ~\rm µF$. A voltage source with U_I feeds the circuit at $f_1 = 50~\rm Hz$.

1. Which values does R and L need to have, when the resonance voltage $|\underline{U}_L|$ and $|\underline{U}_C|$ at f_1 shall show the double value of the input voltage U_I?
2. The components of question 1. shall now be used. What would be the value of ${{|\underline{U}_C|} \over {|\underline{U}_I|}} $ for $f_2 = 60~\rm Hz$?

Exercise 6.3.2 Series Resonant Circuit II

A given R-L-C series circuit is fed with a frequency, $20~\%$ larger than the resonance frequency keeping the amplitude of the input voltage constant. In this situation, the circuit shows a current $30~\%$ lower than the maximum current value.

Result

Calculate the Quality $Q = {{1}\over{R}}\sqrt{{{L}\over{C}}}$.

\begin{align*} Q &= {{ \sqrt{{{1}\over{0.7^2}} - 1} }\over{ 1.2 - {{1}\over{1.2}} }} \\ &= 2.782... \\ \rightarrow Q &= 2.78 \end{align*}

Solution

The solution looks hard at first since no insights for the values of R, C, and L are given.

However, it is possible and there are multiple ways to solve it.

What we know

But first, add some more info, which is always true from resonant circuits at the resonant frequency:

1. $\omega_0 = {{1}\over{\sqrt{LC}}}$
2. $X_{C0} = X_{L0}$
3. $Z = \sqrt{R^2 + (X_L - X_C)^2}$, based on the sum of the impedances $\underline{Z}_{\rm eq} = \underline{X}_R + \underline{X}_C + \underline{X}_L$ and the Pythagorean theorem$

From the task, the following is also known.

1. Using “a frequency, $20~\%$ larger than the resonance frequency”:
    1. $f = 1.2 \cdot f_0 $ and
    2. $\omega = 1.2 \cdot \omega_0 $
2. The circuit shows a current $30~\%$ lower than the maximum current value:

1. The maximum current for the series resonant circuit is given for the minimum impedance Z.
   The minimum impedance Z is given at resonance frequency, and is $Z_{\rm min} = R$
2. Therefore: $Z = {{1}\cdot{0.7}} \cdot R$

Solution 2: The fast path

We start with $Z = \sqrt{R^2 + (X_L - X_C)^2}$ for the cases: (1) at the resonant frequency f_0 and (2) at the given frequency $f = 1.2 \cdot f_0 $

\begin{align*} (1): && Z_0 &= R \\ (2): && Z &= \sqrt{R^2 + (X_L - X_C)^2} \\ \end{align*}

In formula (2) the impedance X_L and X_C are:

- $X_L= \omega \cdot L$ and therefore also $X_L = 1.2 \cdot \omega_0 \cdot L = 1.2 \cdot X_{L0}$
- $X_C= {{1}\over {\omega \cdot C}}$ and therefore also $X_C = {{1}\over {1.2 \cdot \omega \cdot C}} = {{1}\over {1.2}} \cdot X_{C0}$

With $X_{C0} = X_{L0}$ we get for (1):

\begin{align*} Z &= \sqrt{R^2 + \left(1.2\cdot X_{L0} - {{1}\over{1.2}} X_{L0} \right)^2} \\ &= \sqrt{R^2 + X_{L0}^2 \cdot \left(1.2 - {{1}\over{1.2}} \right)^2} \\ \end{align*}

Since we know that $Z = {{1}\cdot{0.7}} \cdot R$ and $Z_0 = R$, we can start by dividing (2) by (1):

\begin{align*} {{(2)}\over{(1)}} : && {{Z}\over{Z_0}} &= {{\sqrt{R^2 + (X_L - X_C)^2} }\over{R}} & &| \text{put in the info from before}\\ && {{1}\cdot{0.7}} &= {{\sqrt{R^2 + X_{L0}^2 \cdot \left(1.2 - {{1}\over{1.2}} \right)^2} }\over{R}} & &| (...)^2 \\ && {{1}\cdot{0.7^2}} &= {{ {R^2 + X_{L0}^2 \cdot \left(1.2 - {{1}\over{1.2}} \right)^2} }\over{R^2}} & &| \cdot R^2 \\ && {{1}\cdot{0.7^2}} \cdot R^2 &= {R^2 + X_{L0}^2 \cdot \left(1.2 - {{1}\over{1.2}} \right)^2} & &| - R^2 \\ && ({{1}\cdot{0.7^2}} -1) \cdot R^2 &= { X_{L0}^2 \cdot \left(1.2 - {{1}\over{1.2}} \right)^2} & &| -R^2 \\ \end{align*}