5 Magnetic Circuits

Student Group

First Name	Surname	Matrikel Nr.

Table of Contents

5. Magnetic Circuits 3
5.1 Linear magnetic Circuits 4
Current-carrying Coil 8
Ferrite Core 8
Airgap 8
Magnetic Circuit in a Formula 9
Notice: 10
Application and Limitations of the Circuit Interpretation 11
Notice: 11
Task 5.1.1 Coil on a plastic Core 12
Task 5.1.2 Coil on a ferrite Core with airgap 12
5.2 Non-linear magnetic Circuits 14
5.3 Mutual Induction and Coupling 14
Effect of Coils onto each other 14
Linked Fluxes 15
Magnetic Circuit with 2 Sources 15
Task 5.3.1 Example for magnetic Circuit with two Sources 17
Step 1: Draw the problem as a network 18
Step 2: Calculate the magnetic resistances 18
Step 3: Calculate the magnetic inductances 18
Effects in the electic circuits 19
positive Polarity 20
negative Polarity 20
5.4 Magnetic Energy 21
magnetic Energy of a magnetic Circuit 21
magnetic Energy of a toroid Coil 22
generalized magnetic Energy 22
Application of the magnetic Energy 22
Circuit with linear magnetic Behavior 24

Circuit with nonlinear magnetic Behavior 24
Circuit with magnetic Hysteresis 25
Tasks 25
Task 5.1.4 Application: Shaded Pole Motor 25
Task 5.1.5 Further Tasks 26
Further Information 26
Switch Reluctance Motor 26
Resolver 27

5. Magnetic Circuits

> For this and the following chapter the online Book 'DC Electrical Circuit Analysis - A Practical Approach' is strongly reccommended as reference. In detail this is chapter 10.3 Magnetic Circuits

- Gyrator–capacitor model

In the previous chapters we got accustomed tof the magnetic field. During this path some similarities from the magnetic field to the electric circuit appeared (see figure 1).

Fig. 1: Similarities magnetic Circuit vs electric Circuit

In this chapter we will investigate, how far we come with such an analogy and where it can by practically applied.

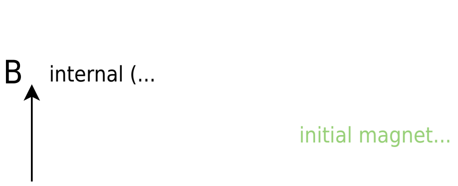
5.1 Linear magnetic Circuits

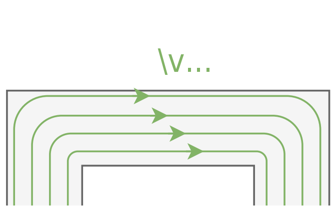
For the upcoming calculations the following assumptions are made

1. The relationship between B and H is linear: $B = \mu \cdot H$
   This is a good estimation when the magnetic field strength lays well below saturation
2. There is not stray field leaking out of the magnetic-field conducting material.
3. The fields inside of airgaps is homogenious. This is true for small airgaps.

One can calulate a lot of simple magnetic circuits, when this assumptions and focusing on the average field line are applied.

Fig. 2: Simplifications and Linearization

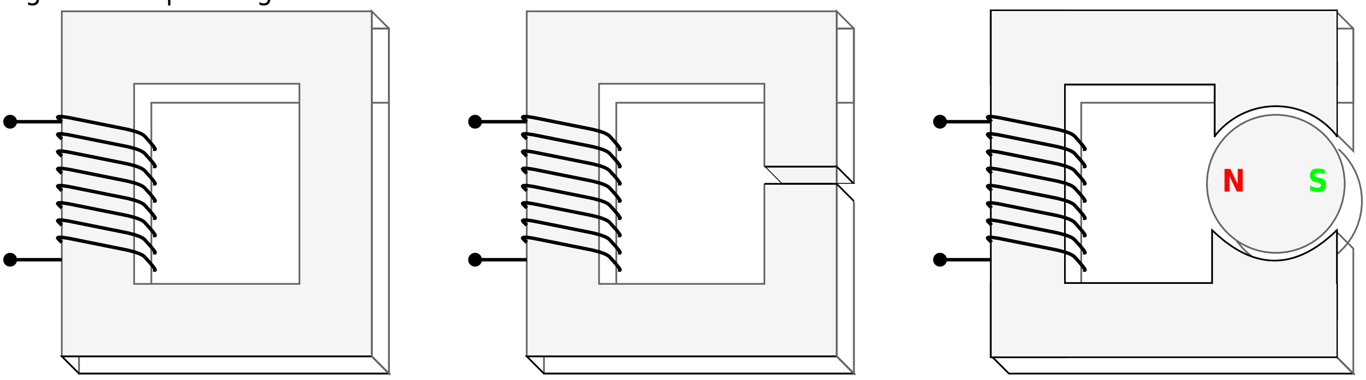

B
internal (...
initial magnet...


\v...

Two simple magnetic circuits are shown in figure 3: They consist of

- a current-carrying coil
- a ferrite core
- an airgap (in picture (2)).

Fig. 3: A simple magnetic Circuit



(1) (2) (3)

Text is not SVG - cannot display

This three parts will be investigated shortly:

Current-carrying Coil

For the magnetic circuit, the coil is parameterized only by:

- its nuber of windings N and
- the passing current i.

This parameters lead to the magnetic voltage $\theta = N\cdot i$.

Ferrite Core

- The core is assumed to be made of ferromagnetic material.
- Therefore, the relative permeability in the core is much larger than in air ($\mu_{r,core}\gg 1$).
- The ferrite core is also filling the inside of the current-carrying coil.
- The ferrite core conducts the magnetic flux around the magnetic circuit (and by this: also to the airgap)

Airgap

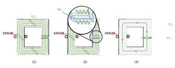
- The airgap interrupts the ferrite core.
- The width of the airgap is small compared to the dimensions of the cross section of the ferrite core.
- The field in the airgap can be used to generate (mechanical) effects within the airgap.
  An example for this can be the force onto a permanent magnet (see figure 3 (3)).

With the above-mentioned assumtions the magnetic flux Φ must remain constant along the ferrite core, so $\Phi_{core}=const.$.
Since the magnetic field lines to neither show sources nor sinks, also the flux passing over to the airgap must be $\Phi_{airgap}=\Phi_{core}=const.$ This can also be seen in figure 4 (1)).

A different view onto this is the closed surface $\vec{A}$ (figure 4 (2)): Based on the examination in Recap of magnetic Field we know that the flux into the volume must be equal the flux out of the volume, or $\Phi_m = \iint_{A} \vec{B} \cdot d \vec{A} = 0$.

The relationship $B=\mu \cdot H$, and $\mu_{core}\gg\mu_{airgap}$ lead to the fact that the ${H}$-Field must be much stronger within the airgap (figure 4 (3)).

Fig. 4: B- and H-field along the ferrite core



Magnetic Circuit in a Formula

Therefore, the following formula is given \begin{align*} \Phi_m &= &\iint_A &\vec{B} \cdot d \vec{A} &= &const. \\ &= & &B \cdot A &= &const. \\ &= & &B_{core} \cdot A_{core} &= &B_{airgap} \cdot A_{airgap} = const. \\ \end{align*}

The assumptions, that there is that the field inside of airgap is homogenious and there is no strayfield lead to the fact, that $A_{core} = A_{airgap}$.
Therefore: \begin{align*} B_{core} &= &B_{airgap} = B \\ \mu_0 \mu_{r,core} H_{core} &= &\mu_0 \mu_{r,airgap} H_{airgap} = {{\Phi}\over{A}} \tag{5.2.1} \end{align*}

Beside this, the magnetic field strength H along one field line is directly given by: \begin{align*} \theta = N \cdot i &= \int_s \vec{H} d\vec{s} = \\ &= \int_{core} \vec{H} d\vec{s} + \int_{airgap} \vec{H} d\vec{s} \\ \end{align*}

With the assumtion of a linear and homogeneous B-Field and the width δ of the airgap, this leads to: \begin{align*} \theta = {H}_{core} l_{core} + {H}_{airgap} \delta \tag{5.2.2} \end{align*}

With the prevoius formula $5.2.1$, this gets to: \begin{align*} \theta &= {{B}\over{\mu_0 \mu_{r,core}}} l_{core} + {{B}\over{\mu_0 \mu_{r,airgap}}} \delta \\ &= {{\Phi \cdot l_{core}}\over{A \cdot \mu_0 \mu_{r,core}}} + {{\Phi \cdot \delta}\over{A \cdot \mu_0 \mu_{r,airgap}}} \\ &= {{1}\over{\mu_0 \mu_{r,core}}}{{l_{core}}\over{A}} \cdot \Phi + {{1}\over{\mu_0 \mu_{r,airgap}}}{{\delta}\over{A}} \cdot \Phi \tag{5.2.3} \end{align*}

Comparing the formula $5.2.3$ with the ohmic resistance and resistivity of two resistors in series, shows something interesting: \begin{align*} u &= &R_1 \cdot &i &+ &R_2 \cdot i \\ &= &\rho {{l_1}\over{A_1}} \cdot &i &+ &\rho {{l_2}\over{A_2}} \cdot i \end{align*}

This leads to:

- The magnetic voltage θ acts like the electic voltage u,

the magnetic flux Φ like the current i.
- The linear relationship $\theta = f(\Phi)$ is also called Hopkinson's Law.

Notice:

- Also for the magnetic circuit one can set up a lumped circuit model (see figure 5).
- Similar to Ohms law, there is a **magnetic resistance** or **reluctance**:
\begin{align*} \boxed{R_m = {{1}\over{\mu_0 \mu_{r}}}{{l}\over{A}}} \end{align*}
- The unit of R_m is $[R_m]= [\theta]/[\Phi] = 1 A / Vs = 1/H $
- The length l is given by the average field line length in the core.
- The Kirchhoff's laws (mesh rule and nodal rule) can also be applied:
  - The sum of the magentic fluxes Φ_i in into a node is: $\sum_i \Phi_i=0 $
  - The sum of the magnetic voltages θ_i along the average field line is: $\sum_i \theta_i=0 $
- The application of the lumped circuit model is based on multiple assumptions.
In contrast to the simplification for the electric current and voltage the simplification for the flux and magnetic voltage in not as exact.

Fig. 5: lumped Circuit Model for magnetic Circuits

Application and Limitations of the Circuit Interpretation

Notice:

"Recipe" for solving magnetic circuits:

1. Watch out for indicidual magnetic resistances in the circuits:
    1. Separate the magnetic circuit into parts, where the the permeability and area of the cross section is constant.
       For these parts B and H is constant. These parts will have individual magnetic resistances.
    2. Each airgap also gest an individual magnetic resistance.
    3. Calculate the magnetic resistance by the average length of field line through the individual parts.
2. Calculate the magnetic circuit as circuit.
    1. Magnetic voltages depicts voltage sources ; magnetic flux depicts currents.
    2. Use the known way to solve the circuit.

> Be aware that the orientation of the current I and the field strength $\vec{H}$ are also connected by the right hant rule.

The results are only allowed as first approximation. The following table shows the contrast between the electric and magnetic field:

Property	**Electric Field**	**Magnetic Field**
Materials	There are “pure” isolator, which are completely non conductive.	All materials have a permeability >0
Sources	The source is concentrated (= there are field sources / electric charges)	The magnetic source (= coil) is distributed
Simplifications	The simplifications often work for good results (small wire diameter, relativly constant resistivity)	The simplification are often too simple (wide spread beyond the average field length, non-linearity of the permeability)

Task 5.1.1 Coil on a plastic Core

A coil is set-up onto a toroidal plastic ring ($\mu_r=1$) with an average circumference of $l_R = 300mm$. The $N=400$ windings are evenly distributed along the circumference. The diameter on the cross section of the plastic ring is $d = 10mm$. In the windings a current of $I=500mA$ is flowing.

Calculate

1. the magnetic field strength H in the middle of the ring cross section.
2. the magnetic flux density B in the middle of the ring cross section.
3. the magnetic resistance R_m of the plastic ring.
4. the magnetic flux Φ.

Task 5.1.2 Coil on a ferrite Core with airgap

The choke coil shown in figure 10 shall be given, with a constant cross section in all legs l_0, l_1, l_2. The number of windings shall be N and the current through a single winding I.

Fig. 10: Example for a Choke Coil

1. Draw the lumped circuit of the magnetic system
2. Calculate all magnetic resistances $R_{m,i}$
3. Calculate the partial fluxes in al the legs of the circuit

5.2 Non-linear magnetic Circuits

not included in the present script

5.3 Mutual Induction and Coupling

Situation: Two coils 1 and 2 near by each other.
Questions:

- Which effect do the coils have onto each other?
- Can we describe the effects with mainly electric properties (i.e. no gemoetric properties)

Fig. 9: Mutual Induction of two Coils

Effect of Coils onto each other

1. Windings N_1 of coil 1 gets feed with current i_1.
2. Coil 1 generates change in flux Φ_{11}
3. Coil 2 gets passed by part of the flux (Φ_{21})
4. Stray flux Φ_{S1} takes not part in coupling

- $\rightarrow$ Coils are magnetically coupled:
  Flux Φ_{11} of coil 1 gets divided into flux Φ_{21} in coil 2 and stray flux Φ_{S1} not passing coil 2:

  $\Phi_{11} = \Phi_{21} + \Phi_{S1}$

- Induced voltage in coil 2:

  $ u_{ind,2} = -{{d}\over{dt}}\Psi_{21} $
  $\quad\quad = -N_2 \cdot {{d}\over{dt}}\Phi_{21} $

- Similar effect on coil 1 due to a current i_2 through coil 2:

$\Phi_{22} = \Phi_{12} + \Phi_{S2}$

$ u_{ind,1} = -N_1 \cdot {{d}\over{dt}}\Phi_{12} $

Linked Fluxes

For the single coil we got the relationship between the linked flux Ψ and the current i as: $\Psi = L \cdot i$.
Now the coils also are interacting with each other. This must also be reflected in the relationship $\Psi_1 = f(i_1, i_2)$, $\Psi_2 = f(i_1, i_2)$: \begin{align*} \Psi_1 &= &\Psi_{11} &+ \Psi_{12} \\ \Psi_2 &= &\Psi_{22} &+ \Psi_{21} \\ \\ \Psi_1 &= &L_{11} \cdot i_1 &+ M_{12} \cdot i_2 \\ \Psi_2 &= &L_{22} \cdot i_2 &+ M_{21} \cdot i_1 \\ \end{align*}

With

- L_{11} and L_{22} as the self induction
- M_{12} and M_{21} as the **mutual induction**

The formula can also be described as: \begin{align*} \left(\begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right) = \left(\begin{array}{c} L_{11} & M_{12} \\ M_{21} & L_{22} \end{array} \right) \cdot \left(\begin{array}{c} i_1 \\ i_2 \end{array} \right) \end{align*}

The view onto the magnetic flux is sometimes good, when effects like an acting Lorentz force in at interest. More often the coils are coupling two electric circuits linke in a transformer or a wireless charger. Here, the effect into the circuits is of interest. This can be calculated with the induced electric voltages $u_{ind,1}$ and $u_{ind,2}$ in each circuits. They are given by the formula $u_{ind,x} = -d\Psi_x /dt$:

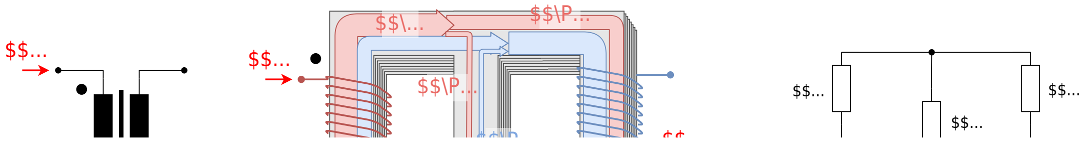
\begin{align*} \left(\begin{array}{c} u_1 \\ u_2 \end{array} \right) &= -{{d}\over{dt}} \left(\begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right) \\ &= -\left(\begin{array}{c} L_{11} & M_{12} \\ M_{21} & L_{22} \end{array} \right) \cdot \left(\begin{array}{c} {{d}\over{dt}}i_1 \\ {{d}\over{dt}}i_2 \end{array} \right) \end{align*}

The main question is now: How do we get L_{11}, M_{12}, L_{22}, M_{21}?

Magnetic Circuit with 2 Sources

In order to get the self induction and mutual induction of two interacting coils, we are going to investigate two coils on an iron core with a middle leg (see figure 10).
Ther the stray flux of the previous situation is only located in the middle leg. This also means, that there is no stray flux outside of the iron core.

Fig. 10: Example for Iron Core with two Coils



The figure 10 shows the fluxes on each parts. The black dots nearby the windings mark the direction of winding:
When there is one current for each winding ingoing into the marked pin, the fluxes will get added up positively.

In order to get L_{11} and L_{22}, we look back to the inductivity L of a long coil with the length l.
This was given in the chapter Self-Induction as

\begin{align*} L &= {{\Psi}\over{i}} \\ L &= \mu_0 \mu_r \cdot N^2 \cdot {{A }\over {l}} \\ \end{align*}

In this case, Ψ was the flux through the coil generated from the coil itself - here L_{11} and L_{22}.
Comparing this formula with the magnetic resistance $R_m = {{1}\over{\mu_0 \mu_{r}}}{{l}\over{A}}$ of the coil, one can conclude:

\begin{align} L = {{N^2 }\over {R_{m,coil}}} \end{align}

Important here is, that we assumed no stray field. When taking this into account for calculating the self induction the magnetic resistance $R_{m,coil}$ ends op to be the resistance of the magnetic circuit, seen by the magnetic voltage source!

\begin{align} \boxed{L = {{N^2 }\over {R_{m1}}}} \end{align}

The magnetic resistance R_{m1} in the example figure 10 is $R_{m1} = R_{m,11} + (R_{m,ss}||R_{m,22})$. Based on this formula, the self inductions L_{11} and L_{22} can be written with the magnetic resistances of the coils as:

\begin{align*} L_{11} &= {{N_1^2 }\over {R_{m1}}} \\ L_{22} &= {{N_2^2 }\over {R_{m2}}} \end{align*}

In order to get the effect of the mutual induction, a coupling coefficient k is introduced. k_{21} describes how much of the flux from coil 1 is acting on coil 2 (similar for k_{12}):

\begin{align*} k_{21} = {{\Phi_{21}}\over{\Phi_{11}}} \\ \end{align*}

When $k_{21}=100\%$, there is no flux in the middle leg but only in the second coil.
For $k_{21}=0\%$ all the flux is in the middle leg circumventing the second coil, i.e. there is no coupling.

The mutual induction M_{21} can be calculated as the fraction of the linked flux Ψ_{11} in coil 2 based on the current i_1 from the coil 1 :

\begin{align*} M_{21} &= {{\Psi_{21}}\over{i_1}} \\ &= {{N_2 \cdot \Phi_{21}}\over{i_1}} \\ &= {{N_2 \cdot k_{21} \cdot \Phi_{11}}\over{i_1}} \\ &= {{N_2}\over{N_1}}k_{21} \cdot {{\Psi_{11}}\over{i_1}} \\ &= {{N_2}\over{N_1}}k_{21} \cdot L_{11} \\ &= {{N_2}\over{N_1}}k_{21} \cdot {{N_1^2 }\over {R_{m1}}} \\ &= k_{21} \cdot {{N_1 \cdot N_2 }\over {R_{m1}}} \\ \end{align*}

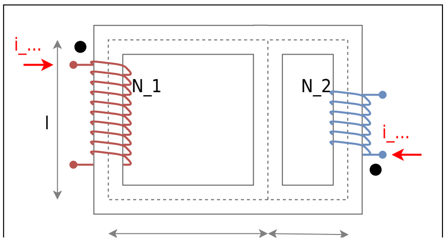
The formula is finally: \begin{align*} \left(\begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right) = \left(\begin{array}{c} {{N_1^2}\over{R_{m1}}} & k_{12}\cdot{{N_1 \cdot N_2}\over{R_{m2}}} \\ k_{21}\cdot{{N_1 \cdot N_2}\over{R_{m1}}} & {{N_2^2}\over{R_{m2}}} \end{array} \right) \cdot \left(\begin{array}{c} i_1 \\ i_2 \end{array} \right) \end{align*}

Task 5.3.1 Example for magnetic Circuit with two Sources

The magnetical configuration in figure 11 shall be given.
The area of the cross-section is $A=9cm^2$ in all parts, the permeability is $\mu_r=800$, the length $l=12cm$ and the number of windings $N_1 = 400$, $N_2=300$. The coupling factors are $k_{12}=0.6$ and $k_{21}=0.8$.

Calculate L_{11}, M_{12}, L_{22}, M_{21}.

Fig. 11: Example for Iron Core with two Coils



Step 1: Draw the problem as a network

Step 2: Calculate the magnetic resistances

The magnetic resistance is summed up looking at the circuit from the source 1: \begin{align*} R_{m1} &= R_{m,11} + R_{m,ss} || R_{m,22} \\ \end{align*}

where the parts are given as \begin{align*} R_{m,11} &= {{1}\over{\mu_0 \mu_{r}}}{{3\cdot l}\over{A}} \\ R_{m,ss} &= {{1}\over{\mu_0 \mu_{r}}}{{1\cdot l}\over{A}} \\ R_{m,22} &= {{1}\over{\mu_0 \mu_{r}}}{{2\cdot l}\over{A}} \\ \end{align*}

With the given geometry this leads to \begin{align*} R_{m1} &= {{1}\over{\mu_0 \mu_{r}}}{{l}\over{A}}\cdot (3 + {{1\cdot 2}\over{1 + 2}}) \\ &= {{1}\over{\mu_0 \mu_{r}}}{{l}\over{A}}\cdot {{11}\over{3}} \\ &= 133 \cdot 10^{3} \cdot {{11}\over{3}} {{1}\over{H}}\\ \end{align*}

Similarly, the magnetic resistance R_{m2} is \begin{align*} R_{m2} &= {{1}\over{\mu_0 \mu_{r}}}{{l}\over{A}}\cdot {{11}\over{4}} \\ &= 133 \cdot 10^{3} \cdot {{11}\over{4}} {{1}\over{H}}\\ \end{align*}

Step 3: Calculate the magnetic inductances

\begin{align*} L_{11} &= {{N_1^2}\over{R_{m1}}} &= 329 mH\\ \\ L_{22} &= {{N_2^2}\over{R_{m2}}} &= 247 mH\\ \\ M_{21} &= k_{21}\cdot{{N_1 \cdot

N_2}\over{R_{m1}}} &= 197 mH\\ \\ M_{12} &= k_{12}\cdot{{N_1 \cdot N_2}\over{R_{m2}}} &= 197 mH\\ \end{align*}

For symmetrical magnetic structures and $\mu_r = const.$ the following applies:

- the mutual inductances are equal: $M_{12}=M_{21}=M$
- the mutual inductance M is: $M = \sqrt{M_{12}\cdot M_{21}} = k \cdot \sqrt {L_{11}\cdot L_{22}}$
- The resulting *total coupling* k is given as \begin{align*} k = \sqrt{k_{12}\cdot k_{21}} \end{align*}

Effects in the electic circuits

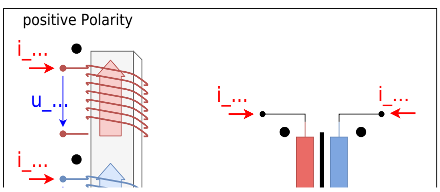
- Whenever two coils are magnetically coupled, not only the self induction L, but also the mutual induction M applies.
- Based on the currents i_1, i_2 in the two circuits, the induced voltages are given by:

\begin{align*} u_1 &= L_{11} \cdot {{di_1}\over{dt}} &+ M \cdot {{di_2}\over{dt}} & \\ u_2 &= M \cdot {{di_1}\over{dt}} &+ L_{22} \cdot {{di_2}\over{dt}} & \\ \end{align*}

It is important to consider the polarity of the fluxes for the calculation in circuits (see figure 15). The **sign of the mudual induction** is influenced by

- the direction of the windings
- the orientation / counting of the current in the circuit

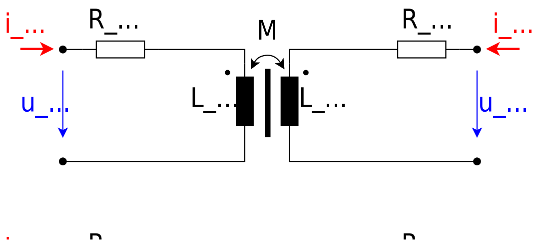
Fig. 15: Polarity of Coupling



positive Polarity

The polarity is positive, when both currents either flowing into or out of the dotted pin (see figure 16).

Fig. 16: Example Circuits with positive Polarity



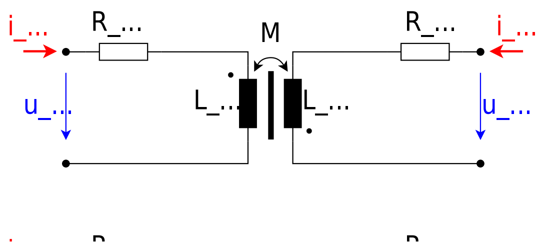
In this case the **mutual induction added positivly**.

The formula of the shown circuitry is then:
$$\begin{align*} u_1 &= R_1 \cdot i_1 &+ L_{11} \cdot {{di_1}\over{dt}} &+ M \cdot {{di_2}\over{dt}} & \\ u_2 &= R_2 \cdot i_2 &+ L_{22} \cdot {{di_2}\over{dt}} &+ M \cdot {{di_1}\over{dt}} & \\ \end{align*}$$

negative Polarity

The polarity is negative, when only one current either flows into the dotted pin and the other one out of the dotted pin (see figure 17).

Fig. 17: Example Circuits with negative Polarity



In this case the **mutual induction added negativly**.

The formula of the shown circuitry is then: $$\begin{align*} u_1 &= R_1 \cdot i_1 &+ L_{11} \cdot {{di_1}\over{dt}} &- M \cdot {{di_2}\over{dt}} & \\ u_2 &= R_2 \cdot i_2 &+ L_{22} \cdot {{di_2}\over{dt}} &- M \cdot {{di_1}\over{dt}} & \\ \end{align*}$$

5.4 Magnetic Energy

The magnetic field of a coil stores magnetic energy. The energy transfer from the electric circuit to the magentic field is also the cause for the “current dampening” effect of the inductor. The energetic turnover for charging an conductor from $i(t_0=0)=0$ to $i(t_1)=I$ is given by:

$$\begin{align*} W_m = \int_0^\infty u(t)\cdot i(t) dt \end{align*}$$

With $u_L = L \cdot di/dt$, this becomes:

$$\begin{align*} W_m &= \int_0^\infty L\cdot {{di}\over{dt}}\cdot i(t) dt \\ &= \int_0^\infty L\cdot i(t) di \\ &= L\cdot \int_0^\infty i(t) di \\ &= L\cdot \left[{{1}\over{2}}i^2\right]_0^I \\ \boxed{W_m = {{1}\over{2}}L\cdot I^2 } \end{align*}$$

magnetic Energy of a magnetic Circuit

With this formula also the stored energy in a macnetic circuit can be calculated. For this, the formula be rewritten by the properties linked flux $\Psi = N\cdot \Phi = L\cdot I$ and magnetic voltage $\theta=N\cdot I = \Phi \cdot R_m$ of the magnetic circuit: $$\begin{align*} \boxed{W_m = {{1}\over{2}}\Psi\cdot I = {{1}\over{2}}{{\Psi^2 }\over{L}}} \end{align*}$$

magnetic Energy of a toroid Coil

The formula can also be used for calculating the stored energy of a toroid coil with N windings, the cross section A and an average length l of a field line. By this, the following formulas can be used: $$\begin{align*} \theta = H\cdot l = N\cdot I \\ \Phi = B \cdot A \end{align*}$$

With the above-mentioned formulas of the magnetic circuit we get: $$\begin{align*} W_m &= {{1}\over{2}}\Psi\cdot I \\ &= {{1}\over{2}}N\cdot B \cdot A \cdot {{H\cdot l}\over{N}} \\ &= {{1}\over{2}}B \cdot H \cdot A \cdot l \\ \end{align*}$$ $$\begin{align*} \boxed{W_m = {{1}\over{2}}B\cdot H \cdot V} \end{align*}$$

The **magnetic energy density** w_m can therefore be calculated as: $$\begin{align*} w_m &= {{W_m}\over{V}} \\ &= {{1}\over{2}}B \cdot H \\ \end{align*}$$

This formula is also true for other types of coils.

generalized magnetic Energy

The general term to find the magnetic energy (e.g. for inhomogenious magnetic fields) is given by $$\begin{align*} W_m &= \iiint_V{w_m dV} \\ &= \iiint_V{\vec{B}\cdot \vec{H} dV} \end{align*}$$

Application of the magnetic Energy

The circuit shown in figure 21 shall now be investigated. The inductor shall be a toroid coil with N windings, the cross section A and an average length l of a field line.

Fig. 21: Example Circuits for calculating the magnetic Energy

The Kirchhoff mesh law leads to:

\begin{align*} u_s = u_R + u_L \\ u_s = R \cdot i + N {{d\Phi}\over{dt}} \end{align*}

Multipling with i and with dt we get the principle of conservation of energy $dw = u \cdot i \cdot dt$ for each small time step.

\begin{align*} u_s \cdot i \cdot dt &= R \cdot i^2 \cdot dt &+ N {{d\Phi}\over{dt}} \cdot i \cdot dt & \\ dW &= dW_R &+ dW_m \end{align*}

On this way we get the magnetic energy as: \begin{align*} dW_m &= N {{d\Phi}\over{dt}} \cdot i \cdot dt \\ W_m &= \int dW_m \\ &= N \int_0^t {{d\Phi}\over{dt}} \cdot i \cdot dt \\ &= N \int_0^{\Phi} i \cdot d\Phi \\ \end{align*}

In a toriod coil with a given cross section A the flux change $d\Phi$ can only be given as a change in the field B. Therefore, $d\Phi = A \cdot dB$. Additionally, we know that the magnetic voltage is given by $\theta(t) = N \cdot i = H(t) \cdot l$. Inluding this into the formula gives us:

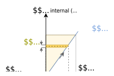
$$\begin{align*} W_m &= N \int_0^{B} i \cdot A \cdot dB \\ &= \int_0^{B} H(B) \cdot l\cdot A \cdot dB \\ &= V\int_0^{B} H(B) \cdot dB \\ \end{align*}$$

We can conclude that the magnetic energy W_m: W_m can be calculated from the H-B-curve by integrating the external magnetic field strength H for each small step of the flux density dB. This wil be shown for the case of a linear magnetic behavior, a nonlinear behavior and the situation with magentic hysteresis shortly.

Circuit with linear magnetic Behavior

In figure 22 the situation for a magnetic material with a linear relationship between B and H is shown. Given by the maximum current I_{max} the maximum field strength H_{max} can be derived. In the circuit in figure 21, the inductor will experience increasing and decreasing current. Therefore, the B-H-curve gets passed through positive and negative values of H and H along the line of $B=\mu H$.

Fig. 22: H-B-Curve for linear material



The situation for integrating the area in the graph is also shown: For each step dB the corresponding value of the field strength H has to be integrated. For $B_0=0$ to $B=B_{max}$ the magnetic energy is

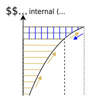
$$\begin{align*} W_m &= V\int_0^{B} H(B) \cdot dB \\ &= V\int_0^{B} {{B}\over{\mu}} \cdot dB \\ &= {{1}\over{2}} V {{B^2}\over{\mu}} \\ &= {{1}\over{2}} V \cdot B \cdot H \\ \end{align*}$$

This situation is a gooed aproximation for air or non-magnetic materials. However, it does not work well for ferrite materials, since they show nonlinear behavior and hysteresis.

Circuit with nonlinear magnetic Behavior

In figure 22 the situation for a magnetic material with a nonlinear relationship between B and H is shown.

Fig. 23: H-B-Curve for nonlinear material

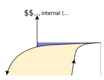


In this case the permeability μ_r is not a constant, but can be represented as a function: $\mu_r= f(B)$. Here, the formula $W_m = V\int_0^{B} H(B) \cdot dB$ also applies - so the magnetic energy is again the area between the curve and the B-axis. As an example the situation of the field strength $H(t_1)=H_1$ is shown. This shall be the field strength after magnetizing the ferrite material to H_{max} (yellow arrows) and then partly demagnetize the material again (blue arrow). The magnetization corresponds to a energy intake to the magnetic field, the demagnetization to an energy outtake.

Moving alon the H-B-curve, one can see, that the energy intake and outtake is the same, when coming back to a start point. This means that the magnetization and demagnetization takes plas lossless in this example. This is a good approximation for magnetically soft materials, however does not work for magnetically hard materials like a permanent magnet. Here, the hysteresis also have to be considered.

Circuit with magnetic Hysteresis

Fig. 24: H-B-Curve material with Hysteresis



Tasks

Task 5.1.4 Application: Shaded Pole Motor

The figure 25 and figure 25 show a shaded pole motor of a commercial oven.

- Find out how this motor works - explicitly: why is there a prefered direction of the motor?
- In which direction does the shown motor run?

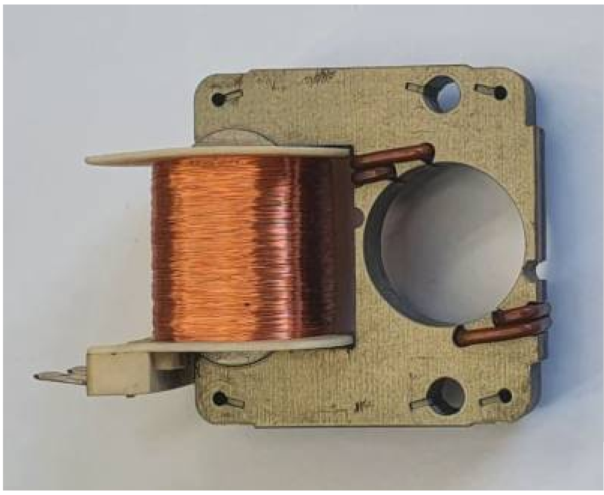


Fig. 25: Core of a shaded Pole Motor

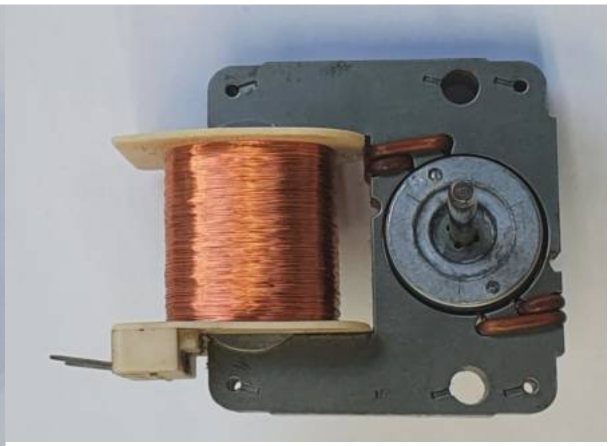


Fig. 26: Setup of the full shaded Pole Motor

Task 5.1.5 Further Tasks

The book DC Electrical Circuit Analysis - A Practical Approach (Fiore) has some nice tasks for beginning in the topic of magnetic circuits

Further Information

Switch Reluctance Motor

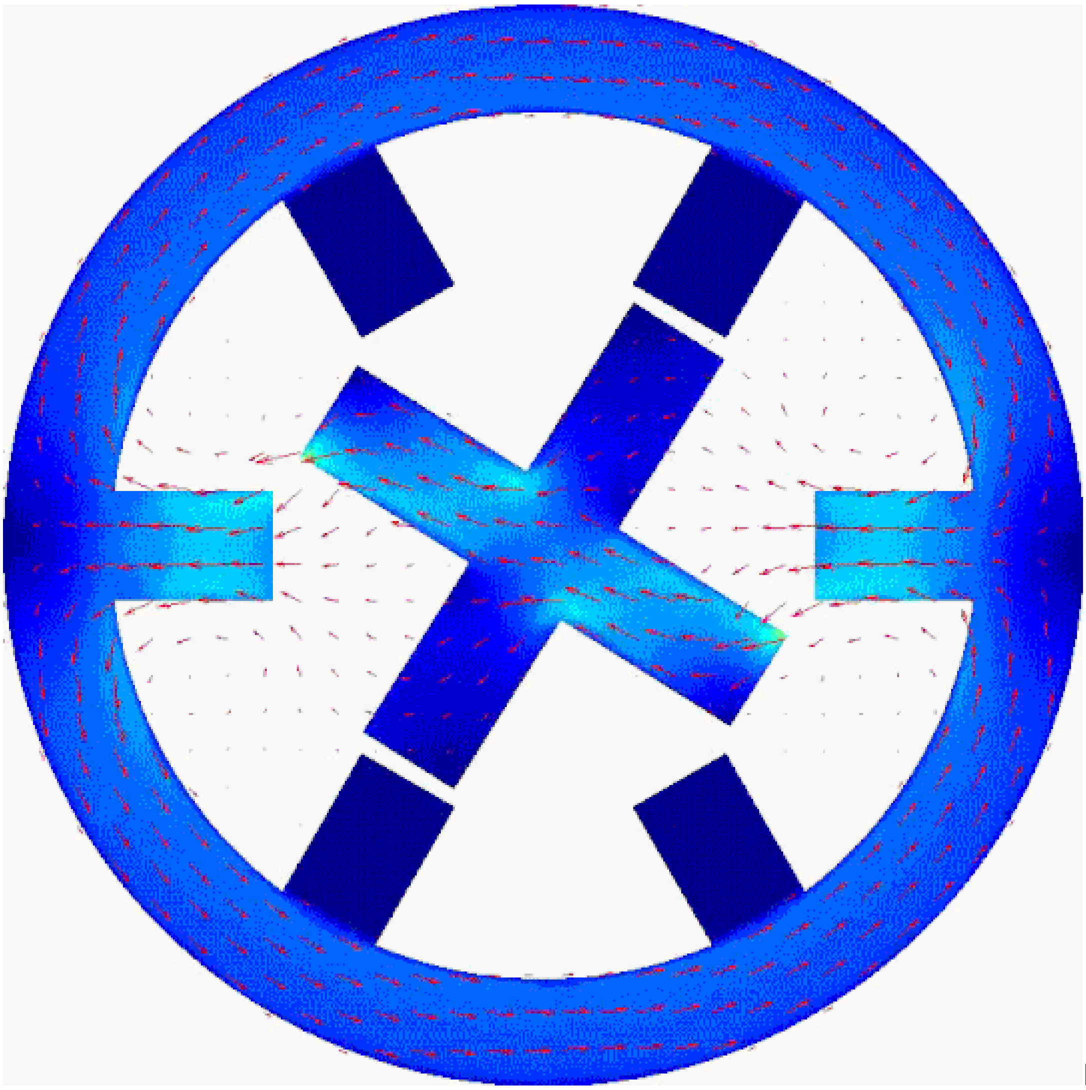


Fig. 28: a switch reluctance Motor as an application for the magnetic resistance

Based on a wikimedia image from Hamidreza D (CC-SA 4.0)

Resolver

Fig. 29: Resolver

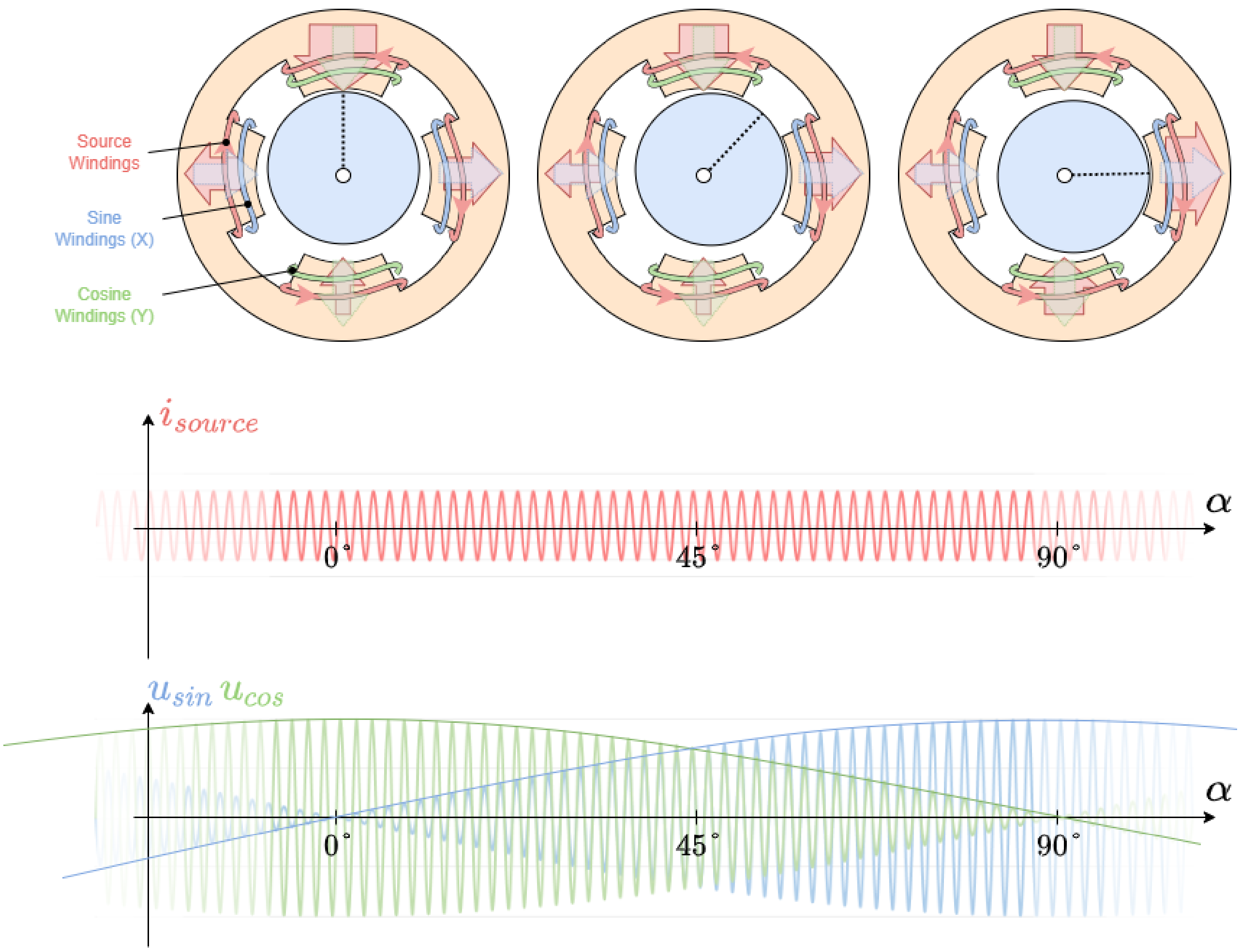


Fig. 30: Overview of the magnetic Formalism