Block 20 — Inductance and Energy

Student Group

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Block 20 — Electromagnetic Induction and Energy

Learning objectives

After this 90-minute block, you can

- …

Preparation at Home

Well, again

- read through the present chapter and write down anything you did not understand.
- Also here, there are some clips for more clarification under 'Embedded resources' (check the text above/below, sometimes only part of the clip is interesting).

For checking your understanding please do the following exercises:

- …

90-minute plan

1. Warm-up (x min):
    1. ….
2. Core concepts & derivations (x min):
    1. …
3. Practice (x min): …
4. Wrap-up (x min): Summary box; common pitfalls checklist.

Conceptual overview

1. …

Core content

Self-Induction

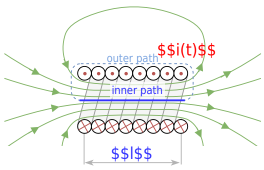
Up to now, we investigated the induction of electric voltages and currents based on the change of an external flux ${\rm d}\Psi / {\rm d}t$. For the induced current $i_{\rm ind}$, we found that it counteracts the change of the external flux (Lenz law).

But what happens, when there is no external field - only a coil which creates the flux change itself (see figure 1)?

Fig. 1: Induction Phenomenons

To understand this, we will investigate the situation for a long coil (figure 2).

Fig. 2: Self-Induction of a Coil



The created field density of the coil can be derived from Ampere's Circuital Law

$$\begin{align*} \theta(t) &= \int & \vec{H}(t) \cdot {\rm d}\vec{s} \\ &= \int & \vec{H}_{\rm inner}(t) \cdot {\rm d}\vec{s} & + & \int \vec{H}_{\rm outer}(t) \cdot {\rm d} \vec{s} \\ &= \int & \vec{H}(t) \cdot {\rm d}\vec{s} & + & 0 \\ &= & {H}(t) \cdot l \\ \end{align*}$$

With magnetic voltage $\theta(t) = N \cdot i$ this lead to the magnetic flux density $B(t)$

$$\begin{align*} N \cdot i &= {H}(t) \cdot l \\ {H}(t) &= {{N \cdot i }\over {l}} \\ {B}(t) &= \mu_0 \mu_{\rm r} \cdot {{N \cdot i }\over {l}} \\ \end{align*}$$

Based on the magnetic flux density $B(t)$ it is possible to calculate the flux $\Phi(t)$:

$$\begin{align*} \Phi(t) &= \iint_A \vec{B}(t) \cdot {\rm d}\vec{A} \\ &= \iint_A \mu_0 \mu_{\rm r} \cdot {{N \cdot i }\over {l}} \cdot {\rm d}A \\ &= \mu_0 \mu_{\rm r} \cdot {{N \cdot i }\over {l}} \cdot A \\ \end{align*}$$

The changing flux Φ is now creating an induced electric voltage and current, which counteracts the initial change of the current. This effect is called **Self Induction**. The induced electric voltage $u_{\rm ind}$ is given by:

\begin{align*} u_{\rm ind} &= - N \cdot {{{\rm d} \Phi(t)}\over{{\rm d}t}} \\ &= - N \cdot {{{\rm d} (\mu_0 \mu_{\rm r} \cdot {{N \cdot i }\over {l}} \cdot A)}\over{{\rm d}t}} \\ &= - N \cdot \mu_0

\mu_{\rm r} \cdot {{N \cdot A }\over {l}} \cdot {{{\rm d}i}\over{{\rm d}t}} \\ \end{align*}

$$\begin{align*} \boxed{ u_{\rm ind} = - \mu_0 \mu_{\rm r} \cdot N^2 \cdot {{A }\over {l}} \cdot {{{\rm d}i}\over{{\rm d}t}} \\ } \\ \text{for a long coil} \end{align*}$$

The result means that the induced electric voltage $u_{\rm ind}$ is proportional to the change of the current ${{\rm d}\over{{\rm d}t}}i$. The proportionality factor is also called **Self-inductance** L (or often simply called inductance).

4.5 Inductance

The inductance is another passive basic component of the electric circuit. Besides the ohmic resistor R and the capacitor C, the inductor L is the lump component entailing the inductance.

Generally, the inductance is defined by: $$\begin{align*} \boxed{ L = \left|{{u_{\rm ind}}\over{{\rm d}i / {\rm d}t}}\right| \\ } \end{align*}$$

The inductance L can also be described differently based on Lenz law $u_{\rm ind} = - {{\rm d}\over{{\rm d}t}}\Psi(t)$:

$$\begin{align*} L &= \left|{{u_{\rm ind}}\over{{\rm d}i / {\rm d}t}}\right| \\ &= {{{d \Psi(t)}/{dt}}\over{{\rm d}i / {\rm d}t}} \\ \end{align*}$$

$$\begin{align*} \boxed{ L = {{ \Psi(t)}\over{i}} } \end{align*}$$

One can also consider an inductor a "conservative person": it does not like to see abrupt changes in the passing current. It reacts to any change in the current with a counteracting voltage since the current change leads to a changing flux and - therefore - an induced voltage. The figure 3 shows an inductor in series with a resistor and a switch (any real switch also behaves as a capacitor, when open). Once the simulation is started, the inductor directly counteracts the current, which is why the current only slowly increases.

The unit of the inductance is $\rm 1 ~Henry = 1 ~H = 1 {{Vs}\over{A}} = 1{{Wb}\over{A}} $

Fig. 3: Example of a Circuit with an Inductor

Mathematically the voltages can be described in the following way:

$$\begin{align*} u_0 &= u_R &+ &u_L \\ &= i \cdot R & + & {{{\rm d}\Psi}\over{{\rm d}t}} \\ &= i \cdot R & + &L \cdot {{{\rm d}i}\over{{\rm d}t}} \\ \end{align*}$$

Inductance of different Components

Long Coil

In the last sub-chapter, the formula of a long coil was already investigated. By these, the inductance of a long coil is

$$\begin{align*} \boxed{L_{\rm long \; coil} = \mu_0 \mu_{\rm r} \cdot N^2 \cdot {{A }\over {l}}} \end{align*}$$

Toroidal Coil

The toroidal coil was analyzed in the last chapter(see magnetic Field Strength Part 1: Toroidal Coil). Here, a rectangular intersection a assumed (see figure 4).

Fig. 4: Self-Induction of a toroidal Coil

This leads to

$$ H(t) = {{N \cdot i}\over {l}} $$

with the mean magnetic path length (= length of the average field line) $l = \pi(r_{\rm o} + r_{\rm i})$:

$$ H(t) = {{N \cdot i}\over { \pi(r_{\rm o} + r_{\rm i})}} $$

The inductance L can be calculated by

\begin{align*} L_{\rm toroidal \; coil} &= {{ \Psi(t)}\over{i}} \\ &= {{ N \cdot \Phi(t)}\over{i}} \\ \end{align*}

With the magnetic flux density $B(t) = \mu_0 \mu_{\rm r} H(t) = \mu_0 \mu_{\rm r} {{i \cdot N }\over {l}}$ and the cross section $A = h (r_{\rm o} - r_{\rm i})$, we get:

\begin{align*} \quad \quad L_{\rm toroidal \; coil} &= {{ N \cdot \mu_0 \mu_{\rm r} {{i \cdot N } \over { \pi(r_{\rm o} + r_{\rm i})}} \cdot h(r_{\rm o} - r_{\rm i})}\over{i}} \\ &= {{ N^2 \cdot \mu_0 \mu_{\rm r} \cdot h(r_{\rm o} - r_{\rm i})}\over{ \pi(r_{\rm o} + r_{\rm i})}} \\ \end{align*}

\begin{align*} \boxed{ L_{\rm toroidal \; coil} = \mu_0 \mu_{\rm r} \cdot N^2 \cdot {{ h(r_{\rm o} - r_{\rm i})}\over{ \pi(r_{\rm o} + r_{\rm i})}} } \end{align*}

6 Inductances in Circuits

Focus here: uncoupled inductors!

Series Circuits

Based on $L = {{ \Psi(t)}\over{i}}$ and Kirchhoff's mesh law ($i=\rm const$) the series circuit of inductions can be interpreted as a single current i which generates multiple linked fluxes Ψ. Since the current must stay constant in the series circuit, the following applies for the equivalent inductor of a series connection of single ones:

\begin{align*} L_{\rm eq} &= {{\sum_i \Psi_i}\over{I}} = \sum_i L_i \end{align*}

A similar result can be derived from the induced voltage $u_{ind}= L {{{\rm d}i}\over{{\rm d}t}}$, when taking the situation of a series circuit (i.e. $i_1 = i_2 = i_1 = ... = i_{\rm eq}$ and $u_{\rm eq}= u_1 + u_2 + ...$):

\begin{align*} & u_{\rm eq} & = &u_1 & + &u_2 &+ ... \\ & L_{\rm eq} {{{\rm d}i_{\rm eq} }\over{{\rm d}t}} & = &L_{1} {{{\rm d}i_{1} }\over{{\rm d}t}} & + &L_{2} {{di_{2} }\over{dt}} &+ ... \\ & L_{\rm eq} {{{\rm d}i }\over{{\rm d}t}} & = &L_{1} {{{\rm d}i }\over{{\rm d}t}} & + &L_{2} {{di }\over{dt}} &+ ... \\ & L_{\rm eq} & = &L_{1} & + &L_{2} &+ ... \\ \end{align*}

Parallel Circuits

For parallel circuits, one can also start with the principles based on Kirchhoff's mesh law:

\begin{align*} u_{\rm eq}= u_1 = u_2 = ... \\ \end{align*}

and Kirchhoff's nodal law:

\begin{align*} i_{\rm eq}= i_1 + i_2 + ... \\ \end{align*}

Here, the formula for the induced voltage has to be rearranged:

\begin{align*} u_{\rm ind} &= L {{{\rm d}i}\over{{\rm d}t}} \quad \quad \quad \quad \bigg| \int(){\rm d}t \\ \int u_{\rm ind} {\rm d}t &= L \cdot i \\ i &= {{1}\over{L}} \cdot \int u_{\rm ind} {\rm d}t \\ \end{align*}

By this, we get:

\begin{align*} i_{\rm eq} &=& i_1 &+& i_2 &+& ... \\ {{1}\over{L_{\rm eq}}} \cdot \int u_{\rm eq}

{\rm d}t &=& {{1}\over{L_1}} \cdot \int u_{1} {\rm d}t &+& {{1}\over{L_2}} \cdot \int u_{2} {\rm d}t &+& ... \\ {{1}\over{L_{\rm eq}}} \cdot \int u {\rm d}t &=& {{1}\over{L_1}} \cdot \int u {\rm d}t &+& {{1}\over{L_2}} \cdot \int u {\rm d}t &+& ... \\ {{1}\over{L_{\rm eq}}} &=& {{1}\over{L_1}} &+& {{1}\over{L_2}} &+& ... \\ \end{align*}

Notice:

The inductor behaves in the parallel and series circuit similar to the resistor.

Common pitfalls

- ...

Exercises

Exercise 4.1.4 Effects of induction I

A change of magnetic flux is passing a coil with a single winding. The following pictures figure ## show different flux-time-diagrams as examples.

- Create for each $\Phi(t)$-diagram the corresponding $u_{\rm ind}(t)$-diagram!
- Write down each maximum value of $u_{\rm ind}(t)$

Fig. ##: Flux-Time-Diagrams

$$\Ph $$\Ph

Solution for (a)

For partwise linear $u_{\rm ind}$ one can derive: \begin{align*} u_{\rm ind} &= -{{{\rm d}\Phi}\over{{\rm d}t}} \\ &= -{{\Delta \Phi}\over{\Delta t}} \end{align*}

For diagram (a):

- $t= 0.0 ... 0.6 ~\rm s$: $u_{\rm ind} = -{{0 ~\rm Vs}\over{0.6 ~\rm s}}= 0$
- $t= 0.6 ... 1.5 ~\rm s$: $u_{\rm ind} = -{{-3.75\cdot 10^{-3} ~\rm Vs}\over{0.9 ~\rm s}}= +4.17 ~\rm mV$
- $t= 1.5 ... 2.1 ~\rm s$: $u_{\rm ind} = -{{0 ~\rm Vs}\over{0.6 ~\rm s}}= 0$

Final result for (a)

Fig. ##: Flux-Time-Diagrams Solution

$$u_{...

Exercise 4.1.5 Effects of induction II

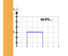
Result for (a)

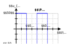
A changing of magnetic flux is passing a coil with a single winding and induces the voltage $u_{\rm ind}(t)$. The following pictures figure ## show different voltage-time diagrams as examples.

- Create for each $u_{\rm ind}(t)$-diagram the corresponding $\Phi(t)$-diagram!
- Write down each maximum value of $\Phi(t)$

Note the given start value Φ_0 for each flux.

Fig. ##: Voltage-Time-Diagrams





Solution for (a)

For partwise linear $u_{\rm ind}$ one can derive: \begin{align*} u_{\rm ind} &= -{{{\rm d}\Phi}\over{{\rm d}t}} \\ \rightarrow \Phi &= -\int_0^t{ u_{\rm ind} \;{\rm d}t} \\ \Phi &= \Phi_0 -\sum_k {u_{{\rm ind},~k} \; \Delta t} \\ \end{align*}

For diagram (a):

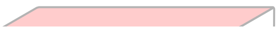
- $t= 0.00 ... 0.04 ~\rm s\quad$: $\quad \Phi = \Phi_0 - {0 \cdot \; \Delta t} \quad\quad\quad\quad\quad\quad\quad= 0 ~\rm Wb$
- $t= 0.04 ... 0.10 ~\rm s\quad$: $\quad \Phi = 0 {~\rm Wb} - {{30 ~\rm mV} \cdot \; (t - 0.04 ~\rm s)} = \quad {1.2 ~\rm mWb} - t \cdot 30 ~\rm mV$
- $t= 0.10 ... 0.14 ~\rm s\quad$: $\quad \Phi = {1.2 ~\rm mWb} - {0.10 ~\rm s} \cdot 30 ~\rm mV \quad = - {1.8 ~\rm mWb}$

Exercise 4.1.6 Coil in magnetic Field I

A single winding is located in a homogenous magnetic field ($B = 0.5 ~\rm T$) between the pole pieces. The winding has a length of $150 ~\rm mm$ and a distance between the conductors of $50 ~\rm mm$ (see figure ##).

- Determine the function $u_{\rm ind}(t)$, when the coil is rotating with $3000 ~\rm min^{-1}$.
- Given a current of $20 ~\rm A$ through the winding: What is the torque $M(\varphi)$ depending on the angle between the surface vector of the winding and the magnetic field?

Fig. ##: Winding between Pole Pieces



Solution

$$\begin{align*} u_{\rm ind} &= - {{{\rm d}\Phi}\over{{\rm d}t}} \\ &= - {{\rm d}\over{{\rm d}t}} B \cdot A \\ &= - B \cdot {{\rm d}\over{{\rm d}t}} A\\ &= - B \cdot {{\rm d}\over{{\rm d}t}} \left(l \cdot d \cdot \cos(\omega t) \right)\\ &= + B \cdot l \cdot d \cdot \omega \cdot \sin(\omega t)\\ \end{align*}$$

Exercise 4.1.7 Coil in magnetic Field II

A rectangular coil is given by the sizes $a=10 ~\rm cm$, $b=4 ~\rm cm$, and the number of windings $N=200$. This coil moves with a constant speed of $v=2 ~\rm m/s$ perpendicular to a homogeneous magnetic field ($B=1.3 ~\rm T$ on a length of $l=5 ~\rm cm$). The area of the coil is tilted with regard to the field in $\alpha=60°$ and enters the field from the left side (see figure ##).

- Determine the function $u_{\rm ind}(t)$ on the coil along the given path. Sketch of the $u_{\rm ind}(t)$ diagram.
- What is the maximum induced voltage $u_{\rm ind,Max}$?

Fig. ##: Winding between Pole Pieces

Solution

Let assume, that l is in the x-axis, $\vec{B}$ in the y-axis and a.

Step 1: Calculate the effective area, perpendicular to the $\vec{B}$-field (independent from whether the area is in the $\vec{B}$-field or not).

For this b has to be projected onto the plane perpendicular to the $\vec{B}$-field: $b_{\rm eff}= b \cdot \cos \alpha$
$$\begin{align*} A_{\rm eff} &= a \cdot b \cdot \cos \alpha \end{align*}$$

Step 2: Focus on entering and exiting the $\vec{B}$-field.
Induction only occurs for ${{\rm d}\over{{\rm d}t}}(A\cdot B)\neq 0$, so here: when the area $A_{\rm eff}$ enters and leave the constant $\vec{B}$-field.

When entering the $\vec{B}$-field the area A with $0<A<A_{eff}$ is in the field. The area moves with v. Therefore, after $\Delta t = b_{\rm eff} \cdot v$ the full $\vec{B}$-field is provided onto the area $A_{\rm eff}$:
$$\begin{align*} u_{\rm ind} &= - {{{\rm d}\Psi}\over{{\rm d}t}} \\ &= -N \cdot {{\rm d}\over{{\rm d}t}} B \cdot A \\ &= -N \cdot {{1}\over{\Delta t}} B \cdot A_{\rm eff} \\ &= -N \cdot {{1}\over{b \cdot \cos \alpha \cdot v}} B \cdot a \cdot b \cdot \cos \alpha \\ &= -N \cdot B \cdot {{a}\over{v}}\\ \end{align*}$$

The following diagram shows …

- … how one can derive the effective width $b_{\rm eff}$, which is projected onto the plane perpendicular to the $\vec{B}$-field: $b_{\rm eff}= b \cdot \cos \alpha$
- … what happens on the effective area $A_{\rm eff}$: there is a change of the field lines in the area only for entering and leaving the $\vec{B}$-field.
- … how the $u_{\rm ind}(t)$ looks as a graph: the part of $A_{\rm eff}$, where the $\vec{B}$-field passes through increase linearly due to the constant speed v

Be aware, that the task did not provide a clue for the direction of windings and therefore it provides no clue for the polarization of the induced voltage.
So, the course of the voltage when entering or exiting is not uniquely given.

Fig. ##: Solution

Exercise E14 Self-Induction
(written test, approx. 8 % of a 120-minute written test, SS2024)

A coil with a length of $30 ~\rm cm$ and a radius of $2 ~\rm cm$ has 500 turns.
The current through the coil changes linearly from 0 to $3 ~\rm A$ in $0.02 ~\rm ms$.
The arrangement is located in air ($\mu_{\rm r}=1$).

1. Calculate the (self-)inductance of the coil.

Path

$\mu_0= 4\pi \cdot 10^{-7} ~\rm Vs/Am$

Result

$L = 1.32 ~\rm mH$

2. Determine the induced voltage in the coil during the change in current.

Path

For the linear change of the current the formula of the induced voltage can also be linearized:
$$\begin{align*} u_{\rm ind} &= - L \cdot {{ {\rm d} i }\over{ {\rm d} t }} \\ &\rightarrow - L \cdot {{ {\Delta} i }\over{ {\Delta} t }} \\ &= - 1.32 \cdot 10^{-3} \cdot {{3 A}\over{0.02 \cdot 10^{-3} s}} \end{align*}$$

Result

$u_{\rm ind} = -197 ~\rm V$

The formula for the induction of a long coil is: \begin{align*} L &= \mu_0 \mu_{\rm r} \cdot N^2 \cdot {{A}\over{l}} \\ &= 4\pi \cdot 10^{-7} {~\rm Vs/Am} \cdot (500)^2 \cdot {{\pi \cdot (2\cdot 10^{-2} ~\rm m)^2}\over{ 2 \cdot 10^{-2} ~\rm m}} \\ \end{align*}

Exercise E8 Self Induction
(written test, approx. 8 % of a 120-minute written test, SS2022)

Accidentally, a motor with an inductance of $L=50 ~\rm \mu H$ is connected to a DC voltage source, which is fused with a circuit breaker.
The circuit breaker opens the circuit at $t_0=0$ with a current of $I=63 ~\rm A$. The current is reduced linearly down to $0 ~\rm A$ within $1 ~\rm \mu s$.
(The inner resistance of the motor shall be neglected.)

2. What is the maximum magnitude of the voltage, which the circuit breaker has to withstand?
Result
Sketch the diagrams $i(t)$ and $u_{\rm ind}(t)$ with the calculated value of the induced voltage.

$u_{\rm ind} (t) = 3150 ~\rm V$

Path

1. Draw the circuit (the circuit breaker can be drawn as a switch), with all voltage and current arrows.

For the maximum voltage on the circuit breaker one has to consider the following:

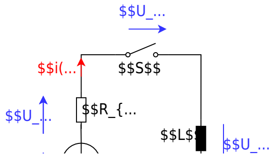
Result

- external voltage of the voltage source $U_\rm s$
- voltage $u_{\rm ind} (t)$ induced by the change of the current

The first one is not given in the exercise, and therefore not considered here.

The induced voltage can be calculated by linearizing the following: \begin{align*} u_{\rm ind} (t) &= - L {{ {\rm d} i}\over{ {\rm d} t }}\\ \rightarrow u_{\rm ind} (t) &= - L {{ \Delta i}\over{ \Delta t }}\\ \end{align*}

With the given details: \begin{align*} u_{\rm ind} (t) &= - L {{ 0 - I}\over{ t_1 - t_0 }}\\ &= 50 \cdot 10^{-6} {~\rm H} {{ 63 ~\rm A}\over{ 1 \cdot 10^{-6} ~\rm s}}\\ &= 3150 {{ ~\rm Vs}\over{ ~\rm A}} \; {{ ~\rm A}\over{ ~\rm s}}\\ \end{align*}



Exercise 4.5.1 Self Induction I

[illegible] core ($\mu_{\rm r}=1000$)

Result

1. Cylindrical long air coil with $N=390$, winding diameter $d=3.0 ~\rm cm$ and length $l=18 ~\rm cm$

Solution

Solution

Solution

[illegible] \end{align*}

[illegible] \\ \end{align*}

\begin{align*} L_1 &= \mu_0 \mu_{\rm r} \cdot N^2 \cdot {{A }\over {l}} \\ &= 4\pi \cdot 10^{-7} {\rm {{H}\over{m}}} \cdot 1 \cdot (390)^2 \cdot {{\pi \cdot (0.03~\rm m)^2 }\over {0.18 ~\rm m}} \end{align*}

Exercise 4.5.2 Self Induction II

A cylindrical air coil (length $l=40 ~\rm cm$, diameter $d=5.0 ~\rm cm$, and a number of windings $N=300$) passes a current of $30 ~\rm A$. The current shall be reduced linearly in $2.0 ~\rm ms$ down to $0.0 ~\rm A$.

Result

What is the amount of the induced voltage $u_{\rm ind}$?

\begin{align*} \left|u_{\rm ind}\right| &= 33 ~\rm V\end{align*}

Solution

The requested induced voltage can be derived by:

\begin{align*} L &= \left|{{u_{\rm ind}}\over{{\rm d}i / {\rm d}t}}\right| \\ \rightarrow \left|u_{\rm ind}\right| &= L \cdot \left|{{{\rm d}i}\over{{\rm d}t}}\right| \\ &= L \cdot \left|{{\Delta i}\over{\Delta t}}\right| \\ \end{align*}

Therefore, we just need the inductance L, since ${{\Delta i}\over{\Delta t}}$ is defined as $30 ~\rm A$ per $2 ~\rm ms$:

\begin{align*} L &= \mu_0 \mu_{\rm r} \cdot N^2 \cdot {{A }\over {l}} \\ \end{align*}

So, the result can be derived as: \begin{align*} \left|u_{\rm ind}\right| &= \mu_0 \mu_{\rm r} \cdot N^2 \cdot {{A }\over {l}} \cdot \left|{{\Delta i}\over{\Delta t}}\right| \\ &= 4\pi \cdot 10^{-7} {\rm {{H}\over{m}}} \cdot 1 \cdot (300)^2 \cdot

{{\pi \cdot (0.05~\rm m)^2 }\over {0.40 ~\rm m}} \cdot {{30 ~\rm A}\over{2 ~\rm ms}} \end{align*}

Exercise 4.5.3 Self Induction III

A coil with the inductance $L=20 ~\rm µH$ passes a current of $40 ~\rm A$. The current shall be reduced linearly in $5 ~\rm µs$ down to $0 ~\rm A$ (see figure 5).

- What is the amount of the induced voltage $u_{\rm ind}$?
- Sketch the course of $u_{\rm ind}(t)$!

Fig. 5: Circuit and timing Diagram

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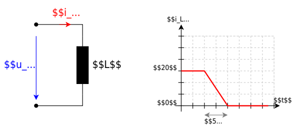

Embedded resources

Explanation (video): ...