Block 03 — Complex Calculus in EE

Learning objectives

After this 90-minute block, you

- know how sine variables can be symbolized by a vector.
- know which parameters can determine a sinusoidal quantity.
- graphically derive a pointer diagram for several existing sine variables.
- can plot the phase shift on the vector and time plots.
- can add sinusoidal quantities in vector and time representation.
- know and apply the impedance of components.
- know the frequency dependence of the impedance of the components. In particular, you should know the effect of the ideal components at very high and very low frequencies and be able to apply it for plausibility checks.
- are able to draw and read pointer diagrams.
- know and apply the complex value formulas of impedance, reactance, and resistance.

Preparation at Home

Well, again

- read through the present chapter and write down anything you did not understand.
- Also here, there are some clips for more clarification under 'Embedded resources' (check the text above/below, sometimes only part of the clip is interesting).

For checking your understanding please do the following exercises:

- …

90-minute plan

1. Warm-up (x min):
    1. ….
2. Core concepts & derivations (x min):
    1. …
3. Practice (x min): …
4. Wrap-up (x min): Summary box; common pitfalls checklist.

Conceptual overview

1. …

Core content

Representation and Interpretation

Up to now, we used for the AC signals the formula $x(t)= \sqrt{2} X \cdot \sin (\omega t + \varphi_x)$ - which was quite obvious.

However, there is an alternative way to look at the alternating sinusoidal signals. For this, we look first at a different, but already a familiar problem (see Abbildung 1).

1. A mechanical, linear spring with the characteristic constant D is displaced due to a mass m in the Earth's gravitational field. The deflection only based on the gravitational field is X_0.
2. At the time $t_0=0$, we deflect this spring a bit more to $X_0 + x(t_0)=X_0 + \hat{X}$ and therefore induce energy into the system.
3. When the mass is released, the mass will spring up and down for $t>0$. The signal can be shown as a shadow when the mass is illuminated sideways.
   For $t>0$, the energy is continuously shifted between potential energy (deflection $x(t)$ around X_0) and kinetic energy (${{\rm d}\over{{\rm d}t}}x(t)$)
4. When looking onto the course of time of $x(t)$, the signal will behave as: $x(t)= \hat{X} \cdot \sin (\omega t + \varphi_x)$
5. The movement of the shadow can also be created by the sideways shadow of a stick on a rotating disc.
   This means, that a two-dimensional rotation is reduced down to a single dimension.

Abb. 1: interpretation of sinusoidal deflection of a spring

The transformation of the two-dimensional rotation to a one-dimensional sinusoidal signal is also shown in Abbildung 2.

Abb. 2: Creation of the sinusoidal signal from a rotational movement

Click on the box „animate?“
press here for the animation

The two-dimensional rotation can be represented with a complex number in Euler's formula. It combines the exponential representation with real part $\Re$ and imaginary part $\Im$ of a complex value: $$ \underline{x}(t)=\hat{X}\cdot {\rm e}^{{\rm j}(\omega t + \varphi_x)} = \Re(\underline{x}) + {\rm j}\cdot \Im(\underline{x})$$

For the imaginary unit ${\rm i}$ the letter ${\rm j}$ is used in electrical engineering since the letter ${\rm i}$ is already taken for currents.

Abb. 3: representation of a phasor on the complex plane

ate System

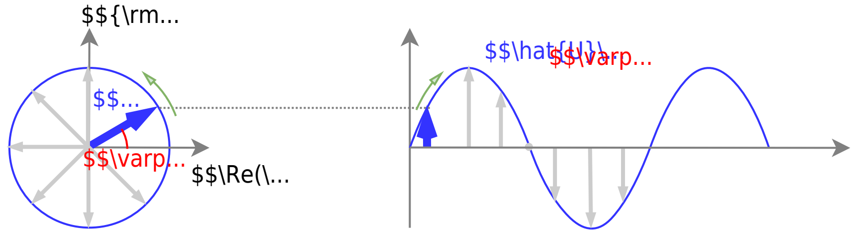
Complex Current and Voltage

The concepts of complex numbers shall now be applied to voltages and currents. Up to now, we used the following formula to represent alternating voltages:

$$u(t)= \sqrt{2} U \cdot \sin (\varphi)$$

This is now interpreted as the instantaneous value of a complex vector $\underline{u}(t)$, which rotates given by the time-dependent angle $\varphi = \omega t + \varphi_u$.

Abb. 4: representation of a voltage phasor on the complex plane



The parts on the complex plane are then given by:

1. The real part $\Re{(\underline{u}(t))} = \sqrt{2}U \cdot \cos (\omega t + \varphi_u)$
2. The imaginary part $\Im{(\underline{u}(t))} = \sqrt{2}U \cdot \sin (\omega t + \varphi_u)$

This is equivalent to the complex phasor $\underline{u}(t)=\sqrt{2}U \cdot {\rm e} ^{{\rm j} (\omega t + \varphi_u)}$

The complex phasor can be separated: \begin{align*} \underline{u}(t) &=\sqrt{2} U \cdot {\rm e}^{{\rm j} (\omega t + \varphi_u)} \\ &=\sqrt{2}\color{blue} {U \cdot {\rm e}^{{\rm j} \varphi_u}} \cdot {\rm e}^{{\rm j} \omega t} \\ &=\sqrt{2}\color{blue}{\underline{U}} \cdot {\rm e}^{{\rm j} \omega t} \\ \end{align*}

The **fixed phasor** (in German: *komplexer Festzeiger*) of the voltage is given by $\color{blue}{\underline{U}}= \color{blue}{U \cdot e ^{j \varphi_u}} $

Generally, from now on not only the voltage will be considered as a phasor, but also the current $\underline{I}$ and derived quantities like the impedance $\underline{X}$.
Therefore, the known properties of complex numbers from Mathematics 101 can be applied:

- A multiplication with j equals a phase shift of $+90°$
- A multiplication with ${{1}\over{j}}$ equals a phase shift of $-90°$

Complex Impedance

Introduction to Complex Impedance

The complex impedance is „nearly" similar calculated like the resistance. In the subchapters before, that impedance Z was calculated by $Z=\frac{U}{I}$.
Now the complex impedance is:

\begin{align*} \underline{Z}&=\frac{\underline{U}}{\underline{I}} \\ &= \Re{(\underline{Z})} + {\rm j} \cdot \Im{(\underline{Z})} \\ &= R + {\rm j} \cdot X \\ &= Z \cdot {\rm e}^{{\rm j} \varphi} \\ &= Z \cdot (\cos \varphi + j \cdot \sin \varphi) \end{align*}

With

- the resistance R (in German: *Widerstand*) as the pure real part
- the reactance X (in German: *Blindwiderstand*) as the pure imaginary part
- the impedance Z (in German: *Scheinwiderstand*) as the complex number given by the complex addition of resistance and the reactance as a complex number

The impedance can be transformed from Cartesian to polar coordinates by:

- $Z=\sqrt{R^2 + X^2}$
- $\varphi = \arctan \frac{X}{R} $

The other way around it is possible to transform by:

- $R = Z \cos \varphi$
- $X = Z \sin \varphi$

Application on pure Loads

With the complex impedance in mind, the Tabelle ## can be expanded to:

Load 		**integral representation **	**complex impedance $\underline{Z}={{\underline{U}}\over{\underline{I}}}$**	**impedance $Z $**	**phase $\varphi $**
Resistance	R	${u} = R \cdot {i}$	$Z_R = R $	$Z_R = R $	$\varphi_R = 0$
Capacitance	C	${u} ={{1}\over{C}}\cdot \int {\rm i} dt$	$Z_C = {{1}\over{{\rm j}\omega \cdot C}} = {{-{\rm j}}\over{\omega \cdot C}}$	$Z_C = {{1}\over{\omega \cdot C}}$	$\varphi_C = -{{1}\over{2}}\pi \hat{=} -90°$
Inductance	L	${u} = L \cdot {{\rm d}\over{{\rm d}t}}{i}$	$Z_L = {\rm j} \omega \cdot L $	$Z_L = \omega \cdot L $	$\varphi_L = +{{1}\over{2}}\pi \hat{=} +90°$

Tab. 1: Formulas for the different pure loads

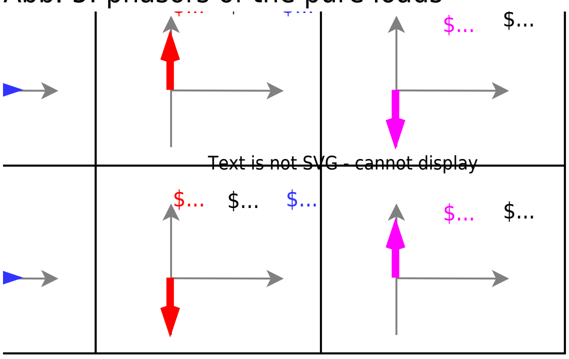
The relationship between ${\rm j}$ and integral calculus should be clear:

1. The derivative of a sinusoidal value - and therefore a phasor - can simply be written as „$\cdot {\rm j}\omega$“,
   which also means a phase shift of $+90°$:
   \begin{align*}{{\rm d}\over{{\rm d}t}} {\rm e}^{{\rm j}(\omega t + \varphi_x)} = {\rm j} \cdot {\rm e}^{{\rm j}(\omega t + \varphi_x)}\end{align*}
2. The integral of a sinusoidal value - and therefore a phasor - can simply be written as „$\cdot (-{{1}\over{ {\rm j}\omega}})$“,
   which also means a phase shift of $-90°$.[1]

   \begin{align*} \int {\rm e}^{{\rm j}(\omega t + \varphi_x)} = {{1}\over{\rm j\omega}} \cdot {\rm e}^{{\rm j}(\omega t + \varphi_x)} = -{{\rm j}\over{\omega}} \cdot {\rm e}^{{\rm j}(\omega t + \varphi_x)} \end{align*}

Once a fixed input voltage is given, the voltage phasor $\underline{U}$, the current phasor $\underline{I}$, and the impedance phasor $\underline{Z}$. In Abbildung 5 these phasors are shown.

Abb. 5: phasors of the pure loads

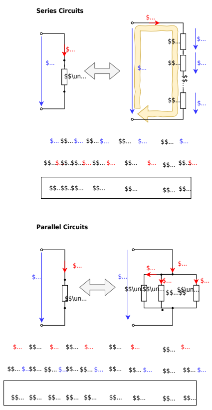


Application on Impedance Networks

Simple Networks

In the chapter Kirchhoff's Circuit Laws we already had a look at simple networks like a series or parallel circuit of resistors.
These formulas not only apply to ohmic resistors but also to impedances:

Abb. 6: Simple Networks

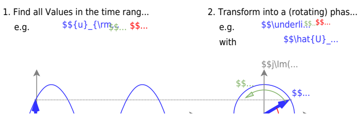


Similarly, the voltage divider, the current divider, the star-delta transformation, and the Thevenin and Northon Theorem can be used, by substituting resistances with impedances. This means for example, every linear source can be represented by an output impedance $\underline{Z}_o$ and an ideal voltage source $\underline{U}$.

More "complex" Networks

For more complex problems having AC values in circuitries, the following approach is beneficial. This concept will be used in the next chapter and in circuit design.

Abb. 7: Approach for AC cicruitries



> **Notice:**
>
> For a complex number are always two values are needed. These are either
>
> 1. the real part (e.g. the resistance) and the imaginary part (e.g. the reactance), or
> 2. the absolute value (e.g. the absolute value of the impedance) and the phase
>
> Therefore, instead of the form $\underline{Z}=Z\cdot {\rm e}^{{\rm j}\varphi}$ for the phasors often the form $Z\angle{\varphi}$ is used.

Common pitfalls

- …

Exercises

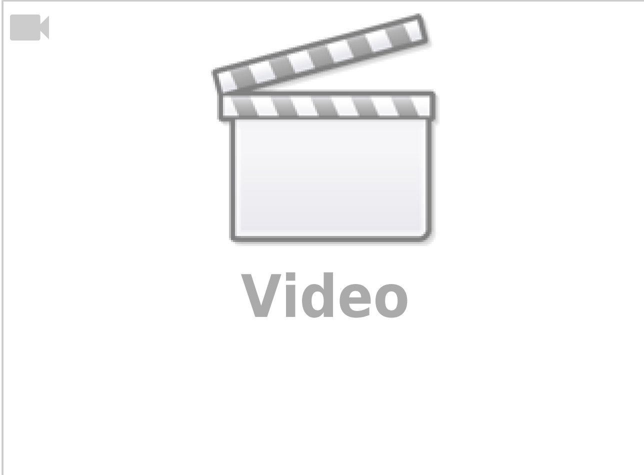
Worked examples

…

Embedded resources

The following two videos explain the basic terms of the complex AC calculus: Impedance, Reactance, Resistance

This does the same [1] in general



, here the integration constant must be considered. This is however often neglectable since only AC values (without a DC value) are considered.

From:
https://wiki.mexle.org/ - **MEXLE Wiki**

Permanent link:
https://wiki.mexle.org/electrical_engineering_and_electronics_2/block03?rev=1772676655

Last update: **2026/03/05 03:10**