Block 03 — Sinusoidal quantities and L, C in AC

Student Group

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Learning objectives

After this 90-minute block, you

- know which types of time-dependent waveforms there are and be able to assign them
- Know the relationship between amplitude and peak-to-peak value.
- Know the relationship between period, frequency, and angular frequency.
- Know the difference between zero phase angle and phase shift angle.
- Know the direction of the phase shift angle.
- know the formula symbols of the above-mentioned quantities.
- calculate the arithmetic mean, the rectified value, and the RMS value.
- know these mean values for sinusoidal quantities.
- know the reason for using the RMS value.
- know that real, lossy components are described by equivalent circuits of ideal components.
- know and be able to apply the definition of apparent resistance, apparent conductance, impedance, and admittance.

Preparation at Home

Well, again

- read through the present chapter and write down anything you did not understand.
- Also here, there are some clips for more clarification under 'Embedded resources' (check the text above/below, sometimes only part of the clip is interesting).

For checking your understanding please do the following exercises:

- …

90-minute plan

1. Warm-up (x min):
    1. ….
2. Core concepts & derivations (x min):
    1. …
3. Practice (x min): …
4. Wrap-up (x min): Summary box; common pitfalls checklist.

Conceptual overview

1. ...

Core content

Introduction to Alternating Current Technology

Up to now, we had analyzed DC signals (chapters 1. - 4.) and abrupt voltage changes for (dis)charging capacitors (chapter 5.). In households, we use alternating voltage (AC) instead of a constant voltage (DC). This is due to at least three main facts

1. Often the voltage given by the **power plant is AC**. This is true for example in all power plants which use electric generators. In these, the mechanical energy of a rotating system is transformed into electric energy using moving magnets, which induce an alternating electric voltage. Some modern plants, like photovoltaic plants, do not primarily generate AC voltages.
2. For long-range power transfer the power losses $P_{\rm loss}$ can be reduced by reducing the currents I since $P_{\rm loss}=R\cdot I^2$. Therefore, for constant power transfer, the voltage has to be increased. This is much easier done with AC voltages: **AC enables the transformation of a lower voltage to a higher** by the use of alternating magnetic fields in a transformer.
3. AC signals have **at least one more value** which can be used for understanding the situation of the source or load. This simplifies the power and load management in a complex power network.

This does not mean that DC power lines are useless or only full of disadvantages:

- A lot of modern loads need DC voltages, like battery-based systems (laptops, electric cars, smartphones). Others can simply be changed into DC loads like systems with electric motors (refrigerators, ovens, lighting, heating).
- Long-range power transfer with DC voltages show often much lower power losses.

Besides the applications in power systems AC values are also important in communication engineering. Acoustic and visual signals like sound and images can often be considered as wavelike AC signals. Additionally, also for signal transfer like Bluetooth, RFID, and antenna design AC signals are important.

To understand these systems a bit more, we will start this chapter with a first introduction to AC systems.

Description of time-dependent Signals

Description of Classification of time-dependent Signals

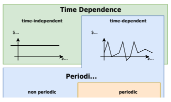
Voltages and currents in the following chapters will be time-dependent values. As already used in chapter 5. for the time-dependent values lowercase letters will be written.

By these time-dependent values, any temporal form of the voltage/current curves is possible (see figure 1).

- We distinguish periodic and non-periodic signals

- One important family of periodic signals is sinusoidal signals
- Sinusoidal signals can be mixed with DC signals

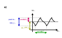
Fig. 1: Classification of time-dependent values



In the following, we will investigate mainly pure AC signals.

Descriptive Values of AC Signals

Fig. 2: Values of AC signals



There are some important characteristic values when investigating AC signals (figure 2). For the signal itself, these are:

- The **DC voltage** or DC offset is given by the value $U_{\rm DC}$ of $V_{\rm DC}$ (in German: Gleichanteil). The DC component also defines the average value of an AC signal.
- The maximum deviation from the DC value is called **peak voltage** $U_\rm p$ (in German : *Spitzespannung*). Specifically for sinusoidal signals the **peak voltage** $U_\rm p$ is also called **amplitude** $\hat{U}$ (in German: *Scheitelwert* or *Amplitude*).
- The voltage difference between maximum and minimum deviation is called **peak-to-peak voltage** $U_{\rm pp}$ (in German: *Spitze-Spitze-Spannung*).
  Be aware, that in English texts the term amplitude is also often used for (non-sinusoidal) $U_{\rm pp}$. Based on German DIN standards the term amplitude is only valid for the sinusoidal peak voltage.

Additionally, there are also characteristic values related to time:

- The shortest time difference for the signal to repeat is called **period** T.
- Based on the period T the frequency $f = {{1}\over{T}}$ can be derived. The unit of the frequency is $1 ~{\rm Hz} = 1 ~\rm Hertz$.
- For calculation, often the **angular frequency** ω is used. The angular frequency is given by $\omega = {{2\pi}\over{T}}$ with the unit ${{1}\over{s}}$.
  The angular frequency represents the angle that is covered in one second.
- Another handy value is the time offset between the start of the sinus wave ($u(t)=0~\rm V$ and rising) and $t=0 ~\rm s$. This difference is often written based on an angular difference

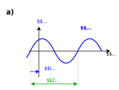
and is called the **phase angle** or **initial phase** φ_U (in German: *Nullphasenwinkel*). This then has to be calculated back to a time value: $\Delta t= {{\varphi_U}\over{\omega}}= \varphi_U\cdot{{T}\over{2\pi}}$

Mathematically, the AC voltages and currents can be written as: $$u(t)=\hat{U}\cdot \sin(\omega t + \varphi_U)$$ $$i(t)=\hat{I}\cdot \sin(\omega t + \varphi_I)$$

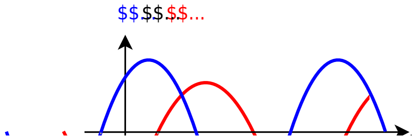
Between the AC voltages and currents, there is also another important characteristic: The **phase difference** $\Delta \varphi$ is given by $\Delta \varphi = \varphi_U - \varphi_I$. The phase difference shows how far the momentary value of the current is ahead of the momentary value of the voltage.

Notice:

The initial phase φ_0 has a direction/sign which has to be considered. In the case **a)** in the picture the zero-crossing of the sinusoidal signal is before $t=0$ or $\omega t =0$. Therefore, the initial phase φ_0 is positive.



Similarly also for the phase difference $\Delta \varphi$ the direction has to be taken into account. In the following image, the zero-crossing of the voltage curve is before the zero-crossing of the current. This leads to a positive phase difference $\Delta \varphi$.



Averaging of AC Signals

To analyze AC signals more, often different types of averages are taken into account. The most important values are:

1. the arithmetic mean $\overline{X}$
2. the rectified value $\overline{|X|}$
3. the RMS value X

The Arithmetic Mean

The arithmetic mean is given by the (equally weighted) averaging of the signed measuring points. For finite values the arithmetic mean is given by: $$\overline{X}={{1}\over{n}}\cdot \sum_{i=1}^n x_i$$

For functions, it is given by: $$\boxed{\overline{X}={{1}\over{T}}\cdot \int_{t=t_0}^{t_0 + T} x(t) {\rm d}t}$$

For pure AC signals, the arithmetic mean is $\overline{X}=0$, since the unsigned value of the integral between the upper half-wave and 0 is equal to the unsigned value of the integral between the lower half-wave and 0.

The Rectified Value

Since the arithmetic mean of pure AC signals with $\overline{X}=0$ does not really give an insight into the signal, different other (weighted) averages can be used.
One of them is the rectified value. For this, the signal is first rectified (visually: negative values are folded up onto the x-axis) and then averaged.
For finite values, the rectified value is given by: $$\overline{|X|}={{1}\over{n}}\cdot \sum_{i=1}^n |x_i|$$

For functions, it is given by: $$\boxed{\overline{|X|}={{1}\over{T}}\cdot \int_{t=t_0}^{t_0 + T} |x(t)| {\rm d}t}$$

> For pure AC signals this results in:
>
> \begin{align*} \overline{|X|} &= {{1}\over{T}}\cdot \int_{t=t_0}^{t_0 + T} |\hat{X}\cdot \sin(\omega t + \varphi) | {\rm d}t \\ \end{align*}
>
> Without limiting the generality, we use $\varphi=0$ and $t_0 = 0$ \begin{align*} \overline{|X|} &= {{1}\over{T}}\cdot \int_{t=0 }^{T } |\hat{X}\cdot \sin(\omega t) | {\rm d}t \\ \end{align*}
>
> Since $sin(\omega t)\geq0$ for $t\in [0,\pi]$, the integral can be changed and the absolute value bars can be excluded like the following
> \begin{align*} \overline{|X|} &= {{1}\over{T}}\cdot 2 \cdot \int_{t=0}^{T/2} \hat{X}\cdot \sin({{2\pi}\over{T}} t) {\rm d}t \\ &= 2 \cdot {{1}\over{T}}\cdot [-\hat{X}\cdot {{T}\over{2\pi}}\cdot \cos({{2\pi}\over{T}} t)]_{t=0}^{T/2} \\ &= 2 \cdot {{1}\over{T}}\cdot {{T}\over{2\pi}}\cdot \hat{X}\cdot [-\cos({{2\pi}\over{T}} t)]_{t=0}^{T/2} \\ &= {{1}\over{\pi}}\cdot \hat{X} \cdot [1+1] \\ \boxed{\overline{|X|} = {{2}\over{\pi}}\cdot \hat{X} \approx 0.6366 \cdot \hat{X}}\\ \end{align*}

The RMS Value

Often it is important to be able to compare AC signals to DC signals by having equivalent values. But what does equivalent mean?
Most importantly, these “equivalent values” are used to compare the output power of a system. One of these equivalent values is the supply voltage value of $230~V$ (or in some countries $110V$). How do we come to these values?

We want to find the voltage U_{DC} and I_{DC} of a DC source, that the output power P_{DC} on a resistor R is similar to the output power P_{AC} of an AC source with the instantaneous values $u(t)$ and $i(t)$. For this, we have to consider the instantaneous power $p(t)$ for a distinct time t and integrate this over one period T.

\begin{align*} P_{\rm DC} &= P_{\rm AC} \\ U_{DC} \cdot I_{\rm DC} &= {{1}\over{T}} \int_{0}^{T} u(t) \cdot i(t) {\rm d}t \\ R \cdot I_{\rm DC}^2 &= {{1}\over{T}} \int_{0}^{T} R \cdot i^2(t) {\rm d}t \\ I_{\rm DC}^2 &= {{1}\over{T}} \int_{0}^{T} i^2(t) {\rm d}t \\ \rightarrow I_{\rm DC} &= \sqrt{{{1}\over{T}} \int_{0}^{T} i^2(t) {\rm d}t} \end{align*}

A similar approach can be used on instantaneous voltage $u(t)$. Generally, the RMS value of X is given by \begin{align*} \boxed{X_{\rm RMS} = \sqrt{{{1}\over{T}} \int_{0}^{T} x^2(t) {\rm d}t}} \end{align*} What is the meaning of RMS? Simple:

By this abbreviation, one can also not forget in which order the formula has to be written… Often the RMS value is also called effective value (in German: Effektivwert).

Note:

- The heat dissipation on a resistor R of an AC current with the RMS value of $I_{\rm RMS} = 1~\rm A$ is equal to the heat dissipation of a DC current with $I_{\rm DC} = 1~\rm A$.
- To shorten writing formulas, the values of AC signals given with uppercase letters will represent the RMS value in the following: $U = U_{\rm RMS}$, $I = I_{\rm RMS}$.
- It holds for AC signals and their RMS values:
    - The resistance is $R={{U}\over{I}}$
    - The power dissipation on a resistor is $P=U\cdot I$

For pure AC signals this results in:

\begin{align*} X &= \sqrt{{{1}\over{T}}\cdot \int_{t=t_0}^{t_0 + T} x^2(t) {\rm d}t} \\ &= \sqrt{{{1}\over{T}}\cdot \int_{t=0} ^{T} \hat{X}^2\cdot \sin^2(\omega t) {\rm d}t} \\ &= \sqrt{{{1}\over{T}}\cdot \int_{t=0} ^{T} \hat{X}^2\cdot {{1}\over{2}}\cdot (1- \cos(2\cdot \omega t)) {\rm d}t} \\ &= \sqrt{{{1}\over{T}}\cdot \hat{X}^2\cdot {{1}\over{2}}\cdot [t + {{1}\over{2\omega }}\cdot \sin(2\cdot \omega t)]_{0}^{T}} \\ &= \sqrt{{{1}\over{T}}\cdot \hat{X}^2\cdot {{1}\over{2}}\cdot (T - 0 + 0 - 0)} \\ &= \sqrt{{{1}\over{2}}\cdot \hat{X}^2} \\ \boxed{X = {{1}\over{\sqrt{2}}}\cdot \hat{X} \approx 0.707\cdot \hat{X}}\\ \end{align*}

Note:

In the following chapters, we will often use for a physical value $x(t)$ a dependency on $\sqrt{2}X$ instead of $\hat{X}$. Therefore, the sinusoidal formula of a physical value x will be : $x(t)=\hat{X}\cdot \sin(\omega t + \varphi_x) \rightarrow x(t)=\sqrt{2}{X}\cdot \sin(\omega t + \varphi_x) $

Comparison of the different Averages

The following simulation shows the different values for averaging a rectangular, a sinusoidal, and a triangular waveform.
Be aware that one has to wait for a full period to see the resulting values on the right outputs of the average generating blocks.

Fig. 3: The averages of different signals

AC Two-Terminal Networks

In the chapters 2. Simple Circuits and 3 Non-ideal Sources and Two-terminal Networks we already have seen, that it is possible to reduce complex circuitries down to equivalent resistors (and ideal sources). This we will try to adopt for AC components, too.

We want to analyze how the relationship between the current through a component and the voltage drop on this component behaves when an AC current is applied.

Resistance

We start with Ohm's law, which states, that the instantaneous voltage $u(t)$ is proportional to the instantaneous current $i(t)$ by the factor R. $$u(t) = R \cdot i(t)$$

Then we insert the functions representing the instantaneous signals: $x(t)= \sqrt{2}{X}\cdot \sin(\omega t + \varphi_x)$: $$\sqrt{2}{U}\cdot \sin(\omega t + \varphi_u) = R \cdot \sqrt{2}{I}\cdot \sin(\omega t + \varphi_i)$$

Since we know, that $u(t)$ must be proportional to $i(t)$ we conclude that for a resistor $\varphi_u=\varphi_i$!

\begin{align*} R &= {{\sqrt{2}{U}\cdot \sin(\omega t + \varphi_i)}\over{\sqrt{2}{I}\cdot \sin(\omega t + \varphi_i) }} \\ &= {{U}\over{I}} \end{align*}

Fig. 4: time course of instantaneous voltage and current on a resistance

This was not too hard and quite obvious. But, what about the other types of passive two-terminal networks - namely the capacitance and inductance?

Capacitance

For the capacitance we have the basic formula: $$C={{Q}\over{U}}$$ This formula is also true for the instantaneous values: $$C={{q(t)}\over{u(t)}}$$ Additionally, we know, that the instantaneous current is defined by $i(t)={{{\rm d}q(t)}\over{{\rm d}t}}$.

By this we can set up the formula: \begin{align*} i(t) &= {{{\rm d}q(t)}\over{{\rm d}t}} \\ &= {{\rm d}\over{{\rm d}t}}\left(C \cdot u(t) \right) \end{align*}

Now, we insert the functions representing the instantaneous signals and calculate the derivative: \begin{align*} \sqrt{2}{I}\cdot \sin(\omega t + \varphi_i) &= {{\rm d}\over{{\rm d}t}}\left(C \cdot \sqrt{2}{U}\cdot \sin(\omega t + \varphi_u) \right) \\ &= C \cdot \sqrt{2}{U}\cdot \omega \cdot \cos(\omega t + \varphi_u) \\ \\ {I}\cdot \sin(\omega t + \varphi_i) &= C \cdot {U}\cdot \omega \cdot \sin(\omega t + \varphi_u + {{1}\over{2}}\pi) \tag{6.3.1} \end{align*}

Equating coefficients in $(6.3.1)$ leads to: \begin{align*} I &= C \cdot U \cdot \omega \\ {{U}\over{I}} &= {{1}\over{\omega \cdot C}} \end{align*} and: \begin{align*} \omega t + \varphi_i &= \omega t + \varphi_u + {{1}\over{2}}\pi \\ \varphi_i &= \varphi_u + {{1}\over{2}}\pi \\ \varphi_u -\varphi_i &= - {{1}\over{2}}\pi \end{align*}

The phase shift of $- {{1}\over{2}}\pi$ can also be seen in figure 6 and figure 5.

Notice:

In order not to mix up the definitions, for AC signals the fraction of RMS voltage by RMS current is called **(apparent) impedance** Z (in German: Scheinwiderstand or Impedanz).
The impedance is generally defined as $$Z = {{U}\over{I}}$$

Only for a pure resistor as a two-terminal network, the impedance Z_R is equal to the value of the resistance: $Z_R=R$.

For the pure capacitive as a two-terminal network, the impedance Z_C is

$Z_C={{1}\over{\omega \cdot C}}$.

Fig. 5: time course of instantaneous voltage and current on a capacitance

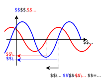


Fig. 6: time course of instantaneous voltage and current on a capacitance

Inductance

The inductance will here be introduced shortly - the detailed introduction is part of electrical engineering 2.
For the capacitance C we had the situation, that it reacts to a voltage change ${{\rm d}\over{{\rm d}t}}u(t)$ with a counteracting current: $$i(t)= C \cdot {{\rm d}\over{{\rm d}t}}u(t)$$ This is due to the fact, that the capacity stores charge carriers q. It appears that "the capacitance does not like voltage changes and reacts with a compensating current". When the voltage on a capacity drops, the capacity supplies a current - when the voltage rises the capacity drains a current.

For an inductance L it is just the other way around: "the inductance does not like current changes and reacts with a compensating voltage drop". Once the current changes the inductance will create a voltage drop that counteracts and continues the current: A current change ${{\rm d}\over{{\rm d}t}}i(t)$ leads to a voltage drop $u(t)$: $$u(t)= L \cdot {{\rm d}\over{{\rm d}t}}i(t)$$ The proportionality factor here is L, the value of the inductance, and it is measured in $[L] = 1~\rm H = 1~Henry$.

We can now again insert the functions representing the instantaneous signals and calculate the derivative: \begin{align*} \sqrt{2}{U}\cdot \sin(\omega t + \varphi_u) &= L \cdot {{\rm d}\over{{\rm d}t}}\left(\sqrt{2}{I}\cdot \sin(\omega t + \varphi_i) \right) \\ &= L \cdot

\sqrt{2}{I}\cdot \omega \cdot \cos(\omega t + \varphi_i) \\ \\ {U}\cdot \sin(\omega t + \varphi_u) &= L \cdot {I}\cdot \omega \cdot \sin(\omega t + \varphi_i + {{1}\over{2}}\pi) \tag{6.3.2} \end{align*}

Equating coefficients in $(6.3.2)$ leads to: \begin{align*} U &= L \cdot {I}\cdot \omega \\ \boxed {Z_L = {{U}\over{I}} = \omega \cdot L} \end{align*} and: \begin{align*} \omega t + \varphi_u &= \omega t + \varphi_i + {{1}\over{2}}\pi \\ \varphi_u &= \varphi_i + {{1}\over{2}}\pi \\ \boxed{\varphi = \varphi_u -\varphi_i = + {{1}\over{2}}\pi } \end{align*}

The phase shift of $+ {{1}\over{2}}\pi$ can also be seen in figure 8 and figure 7.

Fig. 7: time course of instantaneous voltage and current on an inductance

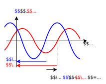


Fig. 8: time course of instantaneous voltage and current on an inductance

Notice:

Remember the formulas for the different pure loads:

Load		**impedance $Z={{U}\over{I}}$**	**phase φ**
Resistance	R	$Z_R = R $	$\varphi_R = 0$
Capacitance	C	$Z_C = {{1}\over{\omega \cdot C}}$	$\varphi_C = - {{1}\over{2}}\pi $
Inductance	L	$Z_L = \omega \cdot L $	$\varphi_L = + {{1}\over{2}}\pi $

Tab. 1: Formulas for the different pure loads

One way to memorize the phase shift is given by the word **CIVIL**:

- **CIVIL**: for a capacitance C the current I leads the voltage V.
  Therefore the phase angle φ_I of the current is larger than the phase angle

φ_U of the voltage: $\rightarrow \varphi = \varphi_U - \varphi_I < 0 $.
- **CIVIL**: for an inductance L the voltage V leads the current I.
  Therefore the phase angle φ_U of the voltge is larger than the phase angle φ_I of the current: $\rightarrow \varphi = \varphi_U - \varphi_I > 0 $.

For the concept of AC two-terminal networks, we are also able to use the DC methods of network analysis to solve AC networks.

Common pitfalls

- …

Exercises

Exercise 6.2.1 The Rectified Value of rectangular and triangular signals

Calculate the rectified value of rectangular and triangular signals! Use similar symmetry simplifications as shown for AC signals. Compare it to the values shown in figure 3.

Exercise 6.3.2 The RMS Value of rectangular and triangular signals

Calculate the RMS value of rectangular and triangular signals! Use similar symmetry simplifications as shown for AC signals. Compare it to the values shown in figure 3.

Exercise 6.3.1 Impedance of single Components I

A coil has a impedance of $80~\Omega$ at a frequency of $500 ~\rm Hz$. At which frequencies the impedance will have the following values?

1. $85 ~\Omega$
2. $120 ~\Omega$
3. $44 ~\Omega$

Solution

When the frequency changes the reactance changes but the inductance is constant.
Therefore, the inductance is needed.
It can be calculated by the given reactance for $f_0 = 500 ~\rm Hz$. \begin{align*} X_{L0}&=2\pi f_0L\\ L &=\frac{X_{L0}}{2\pi f_0} \end{align*}

On the other hand, one can also use the rule of proportion here, and circumvent the calculation of inductance.
It is possible to calculate the reactance at other frequencies with the given reactance. \begin{align*} X_L&=2\pi fL\\ f &=\frac{X_L}{2\pi L}\\ &=\frac{X_L}{X_{L0}}f_0 \end{align*}

With the values given: \begin{equation*} f_1 = \frac{85 ~\Omega}{80~\Omega}\cdot500~{\rm Hz}\qquad f_2 = \frac{120~\Omega}{80~\Omega}\cdot500~{\rm Hz}\qquad f_3 = \frac{44 ~\Omega}{80~\Omega}\cdot500~{\rm Hz} \end{equation*}

Final value

\begin{equation*} f_1=531.25~{\rm Hz}\qquad f_2=750~{\rm Hz}\qquad f_3=275~{\rm Hz} \end{equation*}

Exercise 6.3.2 Impedance of single Components II

A capacitor with $5 ~{\rm µF}$ is connected to a voltage source which generates $U_\sim = 200 ~{\rm V}$. At which frequencies the following currents can be measured?

1. $0.5 ~\rm A$
2. $0.8 ~\rm A$
3. $1.3 ~\rm A$

Exercise 6.3.3 Impedance of single Components III

A capacitor shall have a capacity of $4.7 ~{\rm µF} \pm 10~\%$. This capacitor shall be used with an AC voltage of $400~\rm V$ and $50~\rm Hz$. What is the possible current range which could be found on this component?

Worked examples

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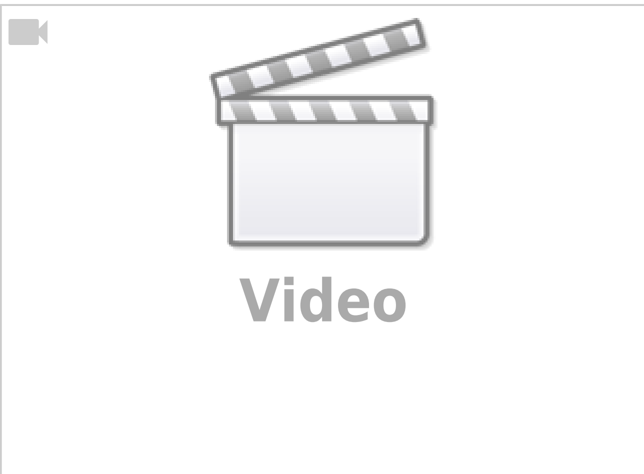
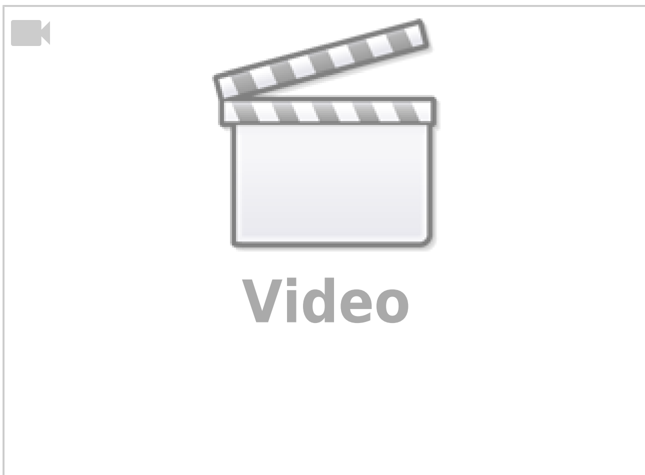
Embedded resources

What's the point of complex numbers?



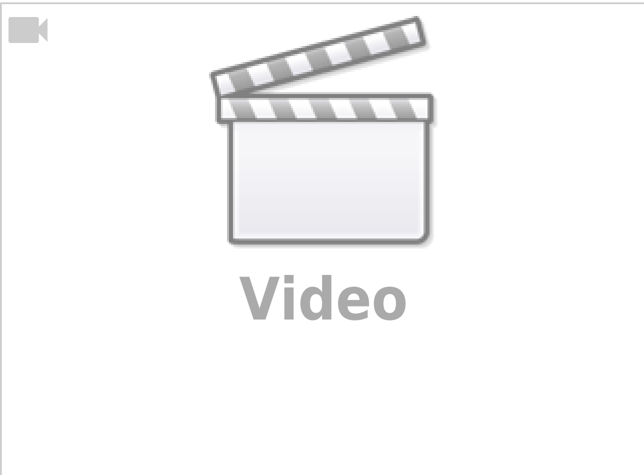
What's the point of complex numbers? (Alternative)



geometric interpretation of the complex multiplication

Why have the absolute values and the angles to be added for multiplication?

The video “Alternating Current AC Basics - Part 1” of EEVblog explains the ideas behind averaged values (arithmetic mean, rectified value and RMS value)



An alternative derivation of the RMS value for a sinusidal function