

Block 06 - Complex Power

Student Group

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Block xx - xxx

Learning objectives

After this 90-minute block, you

1. Know the formula of the instantaneous power of the resistor, inductor, and capacitor and be able to determine its values.

Preparation at Home

Well, again

- read through the present chapter and write down anything you did not understand.
- Also here, there are some clips for more clarification under 'Embedded resources' (check the text above/below, sometimes only part of the clip is interesting).

For checking your understanding please do the following exercises:

- ...

90-minute plan

1. Warm-up (x min):
 1.
2. Core concepts & derivations (x min):
 1. ...
3. Practice (x min): ...
4. Wrap-up (x min): Summary box; common pitfalls checklist.

Conceptual overview

1. ...

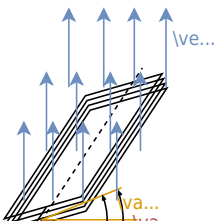
Core content

Last semester in [block18](#) we investigated induction effects have into coils. When the coil is rotating constantly, These can be considered as the AC voltage source on a circuit:

- the sinusoidal alternating voltage is produced by the rotation of a coil in a homogeneous magnetic field, and
- the sinusoidal alternating current is formed by a connected load (or complex impedance).

This will be briefly illustrated here. In [figure 1](#) a coil with w windings is seen in a magnetic field with a magnetic flux density \vec{B} . The coil rotates - starting from φ_0 with angular velocity ω . The rotation changes the chained flux Ψ through the coil and thus a voltage $u(t)$ is induced.

Fig. 1: voltage generation in generator



For the rotation angle φ holds:
$$\begin{aligned} \varphi(t) &= \omega t + \varphi_0 \\ \varphi_0 &= \varphi(t=0) \end{aligned}$$

Thus, the induced voltage $u(t)$ is given by:
$$\begin{aligned} u(t) &= -\frac{d\Psi}{dt} \\ &= -N \cdot \frac{d\Phi}{dt} \\ &= -\hat{\Psi} \cdot \frac{d(\cos(\omega t + \varphi_0))}{dt} \\ &= \omega \hat{\Psi} \cdot \sin(\omega t + \varphi_0) \\ &= \hat{U} \cdot \sin(\omega t + \varphi_0) \end{aligned}$$

Such single-phase systems are therefore alternating current systems, which use one outgoing line and one return line each for the current conduction.

Out of the last formula we derived the following instantaneous voltage $u(t)$
$$u(t) = \hat{U} \cdot \sin(\omega t + \varphi_0) = \sqrt{2} U \cdot \sin(\omega t + \varphi_0)$$

In [block03](#), we used DC power $P = U \cdot I$ and compared it with the instantaneous power $p(t)$ of an AC circuit:

$$p(t) = \color{blue}{u(t)} \cdot \color{red}{i(t)}$$

Additionally we used a bit later in [block03](#) the functions representing the instantaneous signals to derive the inductance: $x(t) = \sqrt{2} X \cdot \sin(\omega t + \varphi_x)$

Now, we will combine both to analyze the AC power on the resistor, capacitor and inductivity in more detail.

Ideal Ohmic resistance R

The simplest component to look at for the instantaneous power is the resistor. For this, we start with the basic definition of the instantaneous voltage $u_R(t)$ (which was given in the last semester) as

$$\color{blue}{u_R(t)} = \sqrt{2} U \sin(\omega t + \varphi_u)$$

With the defining formula for the resistor, we get:

$$\begin{aligned} \color{blue}{u_R(t)} &= R \cdot \color{red}{i(t)} \implies \color{red}{i(t)} = \frac{\color{blue}{u_R(t)}}{R} \\ &= \frac{\sqrt{2} U}{R} \sin(\omega t + \varphi_u) \end{aligned}$$

This leads to an instantaneous power $p_R(t)$ of
$$p_R(t) = \color{blue}{u_R(t)} \cdot \color{red}{i_R(t)} = 2 \cdot \frac{U^2}{R} \sin^2(\omega t + \varphi_u) = \frac{U^2}{R} \left(1 - \cos(2(\omega t + \varphi_u))\right)$$

For the last step the [Double-angle formula](#) " $\cos(2x) = 1 - 2 \sin^2(x)$ " was used.

This result is interesting in the following ways:

1. The part $1 - \cos(2(\omega t + \varphi_u))$ is always non-negative and a shifted sinusoidal function between $0 \dots 2$. The average value of this part is 1 .
2. The average value of $p_R(t)$ is then: $P_R = \frac{U^2}{R}$
3. The use of the $\sqrt{2}$ in the definition $\color{blue}{u_R(t)} = \sqrt{2} U \sin(\omega t + \varphi_u)$ leads to the average power as $P_R = \frac{U^2}{R}$. This formula for the power is exactly like the formula for the power in pure DC situations.

Ideal Inductivity L

A similar approach is done for the ideal inductivity. We again start with the basic definition of the instantaneous voltage

$$u_L(t) = \sqrt{2}U \sin(\omega t + \varphi_u)$$

With the defining formula for inductivity, we get:
$$i_L(t) = \frac{1}{L} \int u_L(t) dt = -\frac{\sqrt{2}}{\omega L} U \cos(\omega t + \varphi_u)$$

This leads to an instantaneous power $p_L(t)$ of

$$p_L(t) = u(t) \cdot i(t) = -2 \frac{U^2}{\omega L} \sin(\omega t + \varphi_u) \cos(\omega t + \varphi_u) = -\frac{U^2}{\omega L} \sin(2\omega t + 2\varphi_u)$$

Again a trigonometric identity ([Double-angle formula](#) “ $\sin(2x) = 2 \sin(x)\cos(x)$ ”) was used.

Also, this result is interesting:

1. The part $\sin(2\omega t + 2\varphi_u)$ has an average value of 0.
2. Therefore, the average value of $p_L(t) = 0$

Ideal Capacity C

Also here, we start with the basic definition of the instantaneous voltage

$$u_C(t) = \sqrt{2}U \sin(\omega t + \varphi_u)$$

With the defining formula for the capacity, we get:
$$i_C(t) = C \frac{du_C(t)}{dt} = \sqrt{2} U \omega C \cos(\omega t + \varphi_u)$$

This leads to an instantaneous power $p_C(t)$ of

$$p_C(t) = u_C(t) \cdot i_C(t) = 2 U^2 \omega C \sin(\omega t + \varphi_u) \cos(\omega t + \varphi_u) = U^2 \omega C \sin(2\omega t + 2\varphi_u)$$

Again this result leads to:

1. The part $\sin(2\omega t + 2\varphi_u)$ has an average value of 0.
2. Therefore, also the average value of $p_C(t) = 0$
 - Instantaneous values of power at R , L , C
 - Active, reactive, apparent, and complex power

This effect can also be seen in the following simulation: The simulation shows three loads, all with an impedance of $|Z| = 1 \sim \Omega$. The diagram on top of each circuit shows the instantaneous **voltage**, **current** and **power**.

1. Ohmic load: The instantaneous voltage is in phase with the instantaneous current. The

instantaneous power is always non-negative. The average power is $P = U^2/R = \frac{1}{2} \hat{U}^2/R = \frac{1}{2} (6V)^2/1 \approx 18 \text{ mW}$

2. Inductive load: The voltage is ahead of the current. The phase angle is $+90^\circ$ (which also reflects the $+j$ in the inductive impedance $+j\omega L$). The instantaneous is half positive, half negative; the average power is zero (in the simulation not completely visible).
3. Capacitive load: The voltage is lagging the current. The phase angle is -90° (which also reflects the $-j$ in the capacitive impedance $\frac{1}{j\omega C}$). The instantaneous is again half positive, half negative; the average power is zero (in the simulation not completely visible).

arbitrary two-terminal Component

For an arbitrary component, we do not have any defining formula. But, the $u(t)$ and $i(t)$ can generally be defined as:

$$\begin{aligned} u(t) &= \sqrt{2}U \sin(\omega t + \varphi_u) \\ i(t) &= \sqrt{2}I \sin(\omega t + \varphi_i) \end{aligned}$$

This leads to an instantaneous power $p(t)$ of

$$p(t) = u(t) \cdot i(t) = 2UI \sin(\omega t + \varphi_u) \sin(\omega t + \varphi_i)$$

The formula can be further simplified with the help of the following equations

- $\varphi = \varphi_u - \varphi_i \quad \rightarrow \varphi_i = \varphi_u - \varphi$
- $\sin(\Box - \varphi) = \sin(\Box) \cos \varphi - \cos(\Box) \sin \varphi$
- $2 \sin \Box \cos \Box = 1 - \cos(2\Box)$
- $2 \sin \Box \sin \Box = \cos(2\Box) - 1$

$$p(t) = UI \left(\cos \varphi \left(1 - \cos(2(\omega t + \varphi_u)) \right) - \sin \varphi \cos(2(\omega t + \varphi_u)) \right)$$

This result is twofold:

1. The part $\cos \varphi \left(1 - \cos(2(\omega t + \varphi_u)) \right)$ results into a non-zero average - explicitly this part is 1 in average. On average the first part of the formula results in $UI \cos \varphi$.
2. The part $-\sin \varphi \cos(2(\omega t + \varphi_u))$ is zero on average, so the second part of the formula results in zero. The amplitude of the second part is $UI \sin \varphi$

Notice:

A distinction is now made between:

- An **active power** (alternatively real or true power, in German: *Wirkleistung*): $P = UI \cos \varphi$
 - The active power represents a pulsed energy drain out of the electrical system (commonly by an ohmic resistor).
 - The active power transforms the electric energy permanently into thermal or

- mechanical energy
- Therefore, the unit of the active power is Watt .
- A **reactive power** (in German: *Blindleistung*): $Q = UI \sin \varphi$
 - The reactive power describes the “sloshing back and forth” of the energy into the electric and/or magnetic fields.
 - The reactive power is completely regained by the electric circuit.
 - To distinguish the values, the unit of the reactive power is VAR (or Var) for **V**olta **r** eactive.
- An **apparent power** (in German *Scheinleistung*): $S = UI$
 - The apparent power is the simple multiplication of the RMS values from the current and the voltage.
 - The apparent power shows only what seems to be a value of power, but can deviate from usable power when inductors or capacitors are used in the circuit.
 - The unit of the apparent power is VA for **V**olta**a**mpere

Similarly, the currents and voltages can be separated into active, reactive, and apparent values.

Based on the given formulas the three types of power are connected with each other. Since the apparent power is given by $S = U \cdot I$, the active power $P = U \cdot I \cdot \cos \varphi = S \cdot \cos \varphi$ and the reactive power $Q = S \cdot \sin \varphi$, the relationship can be shown in a triangle (see [figure 2](#)).

Fig. 2: Power Triangle of active, reactive and apparent power

Generally, the apparent power can also be interpreted as a complex value:

$$\underline{S} = S \cdot e^{j\varphi} = U \cdot I \cdot e^{j\varphi}$$

Based on the definition of the phase angle $\varphi = \varphi_U - \varphi_I$, this can be divided into:

$$\underline{S} = U \cdot I \cdot e^{j(\varphi_U - \varphi_I)} = \underbrace{U \cdot e^{j\varphi_U}}_{\underline{U}} \cdot \underbrace{I \cdot e^{-j\varphi_I}}_{\underline{I}^*}$$

where \underline{I}^* is the complex conjugated value of \underline{I} .

Notice:

The apparent power \underline{S} is given by:

- $\underline{S} = UI \cdot e^{j\varphi}$
- $\underline{S} = UI \cdot (\cos\varphi + j\sin\varphi)$
- $\underline{S} = P + jQ$
- $\underline{S} = \underline{U} \cdot \underline{I}^*$

The following simulation shows three ohmic-inductive loads, all with an impedance of $|Z| = 1 \text{ k}\Omega$, however with different phase angles φ . The diagram on top of each circuit shows the instantaneous **voltage**, **current** and **power**. Similar to the last simulation, a pure ohmic resistance would consume an average power of $P = U^2/R = \frac{1}{2} \hat{U}^2/R = \frac{1}{2} (6 \text{ V})^2/1 \text{ k}\Omega = 18 \text{ mW}$. The three diagrams shall be discussed shortly.

1. Phase angle $\varphi = 10^\circ$: Nearly all of the impedance is given by the resistance and therefore the real part of the impedance. The instantaneous voltage is nearly in phase with the current. The instantaneous power is almost always larger than zero. The average power with 17.47 mW is about the same as for an ohmic impedance.
2. Phase angle $\varphi = 60^\circ$: It is clearly visible, that instantaneous voltage and current are out of phase. The instantaneous power is often lower than zero. The ohmic resistor has $500 \text{ }\Omega = \frac{1}{2}|Z|$, but does not show half of the voltage! This is because the addition has to respect the complex behavior of the values. The complex part is 90° perpendicular to the real part - so they generate a right-angled triangle. The average power with 9 mW is exactly half of the power for an ohmic impedance since only the resistance provides a way for consuming power permanently.
3. Phase angle $\varphi = 84.28^\circ$: The phase angle is calculated in such a way, that the resistance is only 10% of the amplitude of the impedance $|Z|$. In this case, the load is nearly pure inductive. The instantaneous power is consequently almost half of the time lower than zero. The average power here is also only 10% of the power for a pure ohmic impedance.

The next simulation enables us to play around with the phase angle of an impedance. The circuit on the left side is a bit harder to understand but consists of a resistive (real) impedance and a complex impedance, which are driven by an AC voltage source. All of these components are parameterizable in such a way that the phase angle can be manipulated by the slider on the right side.

In the middle part reflects the time course of:

- The instantaneous power p of the **real part (active power)**, the **imaginary part (reactive power)** and **overall power**.
- The instantaneous **voltage** and **current**.

On the right-hand side, the impedance Phasor is shown (lower diagram). The upper diagram depicts the u - i -diagram, which would be a perfect line for a pure ohmic resistance (since $u_R = R \cdot i_R$) and a circle for a pure complex impedance (since the phase angle of $\pm 90^\circ$ between $u_{L, C}$ and $i_{L, C}$). The simulation is in this part not completely perfect: The pure line and circle are sometimes not reachable.

The following questions can be solved with this simulation:

1. How does the amplitude of the active and reactive instantaneous power change, when the phase angle is changed between $-90^\circ \dots +90^\circ$?
2. What is the phase shift between the active and reactive instantaneous power?

1)

Also, the last simulation shows the relation between the phase angle (here: α) and instantaneous values, like power, voltage, and current.

Fig. 3: Simulation of instantaneous power as a function of phase

Change the phase angle with the slider under α
 press here for the animation

Applications

Power Factor Correction

Cables and components have to conduct the sum of active and reactive currents, but only the active current is used outside of the circuit. Therefore, a common goal is to minimize the reactive part. The technical way to represent this is the **power factor** pf is used.

Notice:

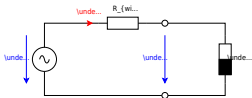
The power factor is given by:

$$\text{pf} = \cos \varphi = \frac{P}{\underline{S}}$$

The power factor shows how much real power one gets out of the needed apparent power.

How does the power factor show the problematic effects? For this one can investigate the situation of an ohmic-inductive load \underline{Z}_L which is connected to a voltage \underline{U}_0 source with a wire R_{wire} . This circuit is shown in [figure 4](#).

Fig. 4: Power Factor of a Power Line



The usable output power is $P_L = U_{\text{L}} \cdot I \cdot \cos \varphi$. Based on this, the current I is:

$$I = \frac{P_L}{U_{\text{L}} \cdot \cos \varphi}$$

The power loss of the wire P_{wire} is therefore:

$$P_{\text{wire}} = R_{\text{wire}} \cdot I^2 = R_{\text{wire}} \cdot \frac{P_L^2}{U_{\text{L}}^2 \cdot \cos^2 \varphi}$$

This means: As smaller, the power factor $\cos \varphi$, as more power losses P_{wire} will be generated. More power losses P_{wire} lead to more heat up to or even beyond the maximum temperature. To compensate for this, the cross-section of the wire has to be increased, which means more copper.

Alternatively, a bad power factor can be compensated with a counteracting complex impedance. This compensating impedance has to provide enough power with the opposite sign to cancel out the unwanted reactive power. The following simulation shows an uncompensated circuit and a circuit with power factor correction. In the latter, the voltage on the load resistor is the same, but the current provided by the power supply is smaller.

Another explanation of the power factor can be seen here:

3. Draw the equivalent series circuit. Calculate the ohmic and inductive impedance and the value of the inductivity.

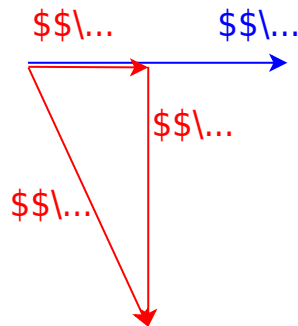
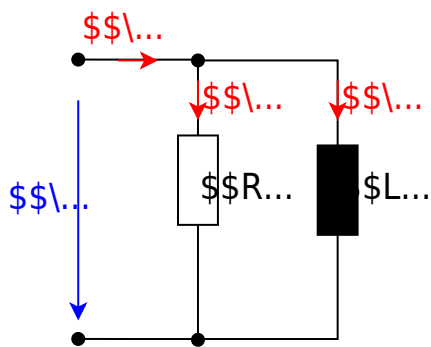
Result for 1.

The real power is
$$P = U \cdot I \cdot \cos \varphi = 115 \text{ V} \cdot 2.6 \text{ A} \cdot 0.3 = 89.7 \text{ W}$$

The reactive power is
$$Q = U \cdot I \cdot \sin \varphi = 115 \text{ V} \cdot 2.6 \text{ A} \cdot \sqrt{1 - 0.3^2} = 285 \text{ Var}$$

The apparent power is
$$S = U \cdot I = 115 \text{ V} \cdot 2.6 \text{ A} = 299 \text{ VA}$$

Result for 2.



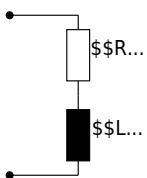
The complex current \underline{I} is given as:

$$\underline{I} = I_R + j I_L \quad \text{and} \quad I = I \cos \varphi - j I \sin \varphi$$

$$\begin{aligned} I_R &= 2.60 \text{ A} \cdot 0.30 = 0.78 \text{ A} \\ I_L &= -2.60 \text{ A} \cdot \sqrt{1 - 0.30^2} = 2.48 \text{ A} \end{aligned}$$

Result for 3.

Important: The cosine function is ambiguous! Based on $\cos \varphi = 0.30$ it is unclear, whether φ is positive or negative. Therefore, only based on the power factor it is unclear whether the circuit is ohmic-inductive or ohmic-capacitive! However, this is explicitly given in the problem definition.



$$\begin{aligned} Z_s &= \frac{U}{I} = \frac{115 \text{ V}}{2.60 \text{ A}} = 44.2 \text{ } \Omega \\ R_s &= \frac{U}{I} \cos \varphi = \frac{115 \text{ V}}{2.60 \text{ A}} \cdot 0.30 = 13.3 \text{ } \Omega \\ X_{Ls} &= \frac{U}{I} \sin \varphi = \frac{115 \text{ V}}{2.60 \text{ A}} \cdot \sqrt{1 - 0.30^2} = 42.2 \text{ } \Omega \\ L_s &= 134 \text{ mH} \end{aligned}$$

Exercise 7.1.3 Power and Power Factor III

A consumer is connected to a $220\text{ V} / 50\text{ Hz}$ network. A current of 20.0 A and a power of 1800 W is measured.

1. What is the value of the active power, the reactive power, and the power factor?
2. Assume that the consumer is a parallel circuit.
 1. Calculate the resistance and reactance.
 2. Calculate the necessary inductance/capacitance.
3. Assume that the consumer is a series circuit.
 1. Calculate the resistance and reactance.
 2. Calculate the necessary inductance/capacitance.

Result for 1.

The active power is $P = 1.80\text{ kW}$.

The apparent power is $S = U \cdot I = 220\text{ V} \cdot 20\text{ A} = 4.40\text{ kVA}$.

The reactive power is $Q = \sqrt{S^2 - P^2} = \sqrt{(4.40\text{ kVA})^2 - (1.80\text{ kW})^2} = 4.01\text{ kVar}$

The power factor is $\cos \varphi = \frac{P}{S} = \frac{1.80\text{ kW}}{4.40\text{ kVA}} = 0.41$.

Result for 2.

Important: The cosine function is ambiguous! Based on $\cos \varphi = 0.30$ it is unclear, whether φ is positive or negative.

Therefore, only based on the power factor it is unclear whether the circuit is ohmic-inductive or ohmic-capacitive!

The consumer is a parallel circuit of the resistance $R\text{ }\Omega$ and the reactance $X\text{ }\Omega$ on the voltage U . Both values can be calculated based on the real and reactive power:

$$\begin{aligned} P &= \frac{U^2}{R\text{ }\Omega} \rightarrow R_p = \frac{U^2}{P} = 26.9\text{ }\Omega \\ Q &= \frac{U^2}{X\text{ }\Omega} \rightarrow X_p = \frac{U^2}{Q} = 12.1\text{ }\Omega \end{aligned}$$

The respective values for inductance/capacitance are: $L = \frac{X_p}{2\pi \cdot f} = 38.4\text{ mH}$ $C = \frac{1}{2\pi \cdot f \cdot X_p} = 263\text{ }\mu\text{F}$

Result for 3.

The consumer is a series circuit of the resistance $R\text{ }\Omega$ and the reactance $X\text{ }\Omega$ with the current I . Both values can be calculated based on the real and reactive power:

$$\begin{aligned} P &= I^2 \cdot R\text{ }\Omega \rightarrow R_s = \frac{P}{I^2} = 4.50\text{ }\Omega \\ Q &= I^2 \cdot X\text{ }\Omega \rightarrow X_s = \frac{Q}{I^2} = 10.0\text{ }\Omega \end{aligned}$$

The respective values for inductance/capacitance are: $L = \frac{X_s}{2\pi \cdot f} = 31.9\text{ mH}$ $C = \frac{1}{2\pi \cdot f \cdot X_s} = 318\text{ }\mu\text{F}$

Exercise 7.1.4 Power and Power Factor IV

An uncompensated ohmic-inductive series circuit shows at $U=230\text{ V}$, $f=50\text{ Hz}$ the current $I_{\text{RL}}=7\text{ A}$, $P_{\text{RL}}=1.3\text{ kW}$

The power factor shall be compensated to $\cos\varphi = 1$ via a parallel compensation.

1. Calculate the apparent power, the reactive power, the phase angle, and the power factor before the compensation.
2. Calculate the capacity C which has to be connected in parallel to get $\cos\varphi=1$.

Solution

$$\begin{aligned} S &= U \cdot I_{\text{RL}} \quad Q = \sqrt{S^2 - P_{\text{RL}}^2} \quad \varphi = \\ \arctan\left(\frac{Q}{P}\right) &= \arccos\left(\frac{P}{S}\right) \end{aligned}$$

The inductor L creates the reactive power $Q = Q_L$. To compensate for a equivalent reactive power $|Q_C| = |Q_L|$ has to be given by a capacitor. The reactive power is given by:
$$Q = \operatorname{Re}(U) \cdot \operatorname{Im}(I) = U \cdot \frac{U}{X} = \frac{U^2}{X}$$

The capacity can therefore be calculated by
$$X_C = \frac{U^2}{Q_L} = \frac{1}{\omega C} \quad \rightarrow \quad C = \frac{1}{\omega U^2}$$

Result

$$\begin{aligned} S &= 1.62\text{ kVA} \quad Q = 0.95\text{ kVAR} \quad \varphi = +36^\circ \quad C = \\ &57.2\text{ }\mu\text{F} \end{aligned}$$

Embedded resources

Explanation (video): ...

1)

For further development: [CalcImpedances](#)
[VarResistor](#) [varLC](#) [SineGenerator](#) [full simu](#)

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