Block 09/10 — Transformers and Magnetic Coupling

Student Group

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Block 09/10 — Transformers and Magnetic Coupling

Learning objectives

After this 90-minute block, you can

- explain how two coils can exchange energy by a common magnetic flux Φ.
- use the ideal transformer equations

$$\begin{align*} \frac{\underline{U}_1^}{\underline{U}_2^}=\frac{N_1}{N_2}=n, \qquad \frac{\underline{I}_1}{\underline{I}_2}=-\frac{1}{n} \end{align*}$$ with a clear sign convention.

- explain mutual inductance M using flux linkage and magnetic reluctance $R_{\rm m}$.
- distinguish **main flux**, **leakage flux**, **copper losses**, and **iron losses** in a real transformer.
- refer secondary-side quantities to the primary side using $\underline{U}'_2=n\underline{U}_2$, $\underline{I}'_2=\frac{1}{n}\underline{I}_2$, $R'_2=n^2R_2$, and $X'_{2\sigma}=n^2X_{2\sigma}$.
- interpret the no-load test and short-circuit test using the reduced equivalent circuit.
- calculate short-circuit voltage $u_{\rm k}$, continuous short-circuit current $I_{\rm 1k}$, and an estimated initial peak short-circuit current.
- connect transformer parameters to engineering applications in mechatronics and robotics, such as isolated power supplies, motor current measurement, welding transformers, and safety transformers.

Preparation at Home

Well, again

- read through the present chapter and write down anything you did not understand.
- Repeat the EEE1 ideas of magnetic flux and induction, magnetic circuits, and inductance and magnetic energy.
- Repeat from EEE2 the use of sinusoidal quantities, complex calculation, and complex power.

For checking your understanding please do the quick checks in the exercise section.

90-minute plan

- **Warm-up (10 min):**
  - Where do transformers occur in robots and automation systems?
  - Recall: Faraday induction from EEE1 — a changing magnetic flux induces a voltage.
  - Recall: in AC analysis we use RMS phasors $\underline{U}$, $\underline{I}$, and

impedances $j\omega L$.

- **Core concepts and derivations (55 min):**
  - Ideal transformer: common flux, voltage ratio, current ratio, power balance.
  - Mutual inductance: how flux from one coil links another coil.
  - Magnetic coupling with reluctance $R_{\rm m}$.
  - Real transformer: winding resistances, leakage inductances, iron-loss resistance.
  - Reduced equivalent circuit: refer secondary quantities to the primary side.
  - No-load and short-circuit operation: what can be measured, what can be neglected.
- **Practice (20 min):**
  - Quick ratio calculations for step-up and step-down transformers.
  - Short-circuit current calculation for a transformer used in an actuator supply.
- **Wrap-up (5 min):**
  - Summary box: ideal transformer, mutual inductance, real transformer, reduced circuit, short-circuit parameters.
  - Common pitfalls checklist.

Conceptual overview

- A transformer is **not** a DC component. It needs a changing magnetic flux. In normal operation this is usually a sinusoidal flux created by AC voltage.
- The transformer does not “create power”. Ideally, it trades voltage for current:

$$\begin{align*} \text{higher voltage} \quad \Longleftrightarrow \quad \text{lower current} \end{align*}$$

- The link between the two windings is the magnetic field in the iron core. This continues directly from EEE1:
  - induction explains why a changing flux induces voltage.
  - magnetic circuits explains why the iron core guides the flux.
  - inductance explains how flux linkage and current are connected.
- Mutual inductance M measures how strongly one coil “notices” the changing current in another coil.
- A real transformer is almost ideal, but not quite:
  - R_1, R_2: copper losses in the windings.
  - $L_{1\sigma}, L_{2\sigma}$: leakage flux that does not couple both windings.
  - $R_{\rm Fe}$: iron losses in the core.
  - $L_{\rm H}$: main magnetizing inductance needed to create the main flux.
- In engineering, transformer data such as $u_{\rm k}$ are not abstract: they determine voltage drop, fault current, thermal stress, and protection design.

Core content

Short Review of the Flux

In EEE1 we considered magnetic flux Φ, flux linkage / linked flux Ψ, and induction.
For one coil with N turns the flux linkage is

$$\begin{align*} \Psi = N\Phi . \end{align*}$$

Faraday's law gives

$$\begin{align*} u(t)=\frac{{\rm d}\Psi}{{\rm d}t}=N\frac{{\rm d}\Phi}{{\rm d}t}. \end{align*}$$

(Be aware of Lenz law: Here u(t) is the terminal voltage according to the chosen voltage reference arrow. The induced voltage $u_{\rm ind}$ according to Faraday–Lenz would have the opposite sign)
In sinusoidal steady state this becomes the phasor equation

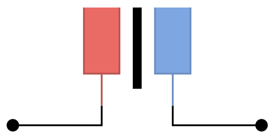
$$\begin{align*} \underline{U}=j\omega\underline{\Psi}=j\omega N\underline{\Phi}. \end{align*}$$

This is the starting point for the transformer.

Polarity and the dot convention

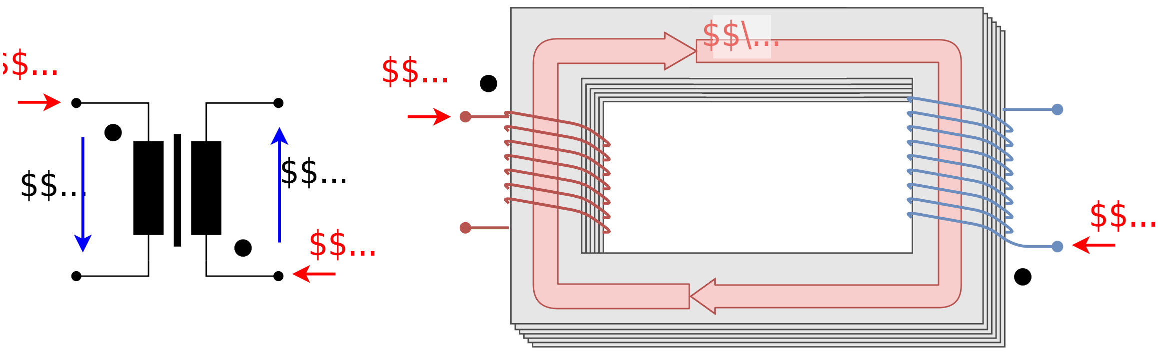
Before we start with the transformer, we have to look at a common convention for the orientation of two coils to each other.

Fig. 1: Dot convention: the dots indicate corresponding winding ends.



> **Rule of thumb**
>
> - If both currents enter dotted terminals, the fluxes support each other.
> - If one current enters a dotted terminal and the other current leaves a dotted terminal, the fluxes oppose each other.

Ideal single-phase transformer



(1) electrically (2) mechanically

For an ideal transformer we assume:

- both windings are linked by the same magnetic flux \(\underline{\Phi} \),
- there is no leakage flux,
- there are no winding resistances,
- there are no iron losses,
- the transformer stores no net energy over one period.

Let \(N_1\) be the number of turns of the primary winding and \(N_2\) the number of turns of the secondary winding.

\[\begin{align*} \underline{\Psi}_1 &= N_1\underline{\Phi}, & \underline{U}_1 &=

j\omega\underline{\Psi}_1 = j\omega N_1\underline{\Phi}, \\ \underline{\Psi}_2 &= N_2\underline{\Phi}, & \underline{U}_2 &= j\omega\underline{\Psi}_2 = j\omega N_2\underline{\Phi}. \end{align*} \]

Dividing the two voltage equations gives the **turns ratio**

\[\begin{align*} \boxed{ \frac{\underline{U}_1^}{\underline{U}_2^} = \frac{N_1}{N_2} = n } \end{align*} \]

with

\[\begin{align*} n=\frac{N_1}{N_2}. \end{align*} \]

Remember: ideal transformer ratios

With the indicated reference arrows and a lossless transformer, the resulting complex power as a sum of input and output power $\underline{S}$ must be zero:

\[\begin{align*} \underline{S}_1+\underline{S}_2= \underline{U}_1\underline{I}^*_1+\underline{U}_2\underline{I}^*_2=0 \end{align*} \]

and therefore

\[\begin{align*} \boxed{ \frac{\underline{I}_1}{\underline{I}_2} = -\frac{\underline{U}_2^}{\underline{U}_1^} = -\frac{N_2}{N_1} = -\frac{1}{n} } \end{align*} \]

The minus sign is not a "loss". It is caused by the chosen current arrows.
The primary side **absorbs** power while the secondary side **delivers** power to the load.

Analogy: gearbox for voltage and current

An ideal transformer behaves like a lossless gearbox:

- a gearbox can trade speed for torque,
- a transformer can trade voltage for current.

For a step-down transformer:

\[\begin{align*} \text{lower voltage} \quad \Longleftrightarrow \quad \text{higher current}. \end{align*} \]

The power is ideally conserved, just as mechanical power is ideally conserved in a lossless gearbox.

Physical interpretation

- If \(N_2<N_1\), the transformer steps the voltage down: \(U_2<U_1\).
- At the same time, the secondary current can be higher: \(I_2>I_1\).
- This is useful in robotics power supplies: a mains-side transformer or isolated converter stage

may reduce voltage while increasing available current for actuators.

Exercise E1 step-down transformer for a robot controller

A transformer has $N_1=800$ turns and $N_2=80$ turns. The primary RMS voltage is $U_1=230~{\rm V}$.

$$\begin{align*} n &= \frac{N_1}{N_2} = \frac{800}{80} =10, \\ U_2 &= \frac{U_1}{n} = \frac{230~{\rm V}}{10} =23~{\rm V}. \end{align*}$$

If the secondary side supplies $I_2=4~{\rm A}$, the ideal primary current magnitude is

$$\begin{align*} I_1 &= \frac{I_2}{n} = \frac{4~{\rm A}}{10} =0.4~{\rm A}. \end{align*}$$

The apparent power is equal on both sides:

$$\begin{align*} S_1 &= U_1I_1=230~{\rm V}\cdot 0.4~{\rm A}=92~{\rm VA}, \\ S_2 &= U_2I_2=23~{\rm V}\cdot 4~{\rm A}=92~{\rm VA}. \end{align*}$$

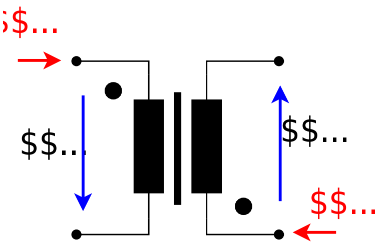
Linked fluxes and mutual inductance

For a single coil we already know that its flux linkage $\Psi = N\Phi$ is proportional to the current i through the coil

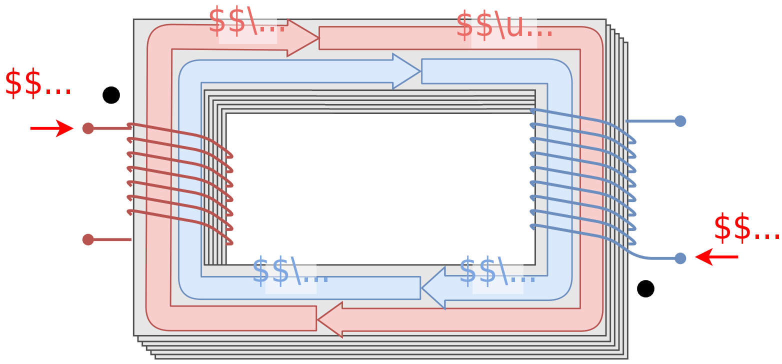
$$\begin{align*} \Psi=L i \end{align*}$$

$$\begin{align*} \underline{\Psi}=L \underline{I} . \end{align*}$$

For two coupled coils 1 and 2, each flux linkage $\underline{\Psi}_1 = N\underline{\Phi}_1$ and $\underline{\Psi}_2 = N\underline{\Phi}_2$ depend on both currents $\underline{I}_1$ and $\underline{I}_2$.



(1) electrically



(2) mechanically

Not only the current through the coil generates a part of the flux linkage, but also the other coil provides a part for the flux linkage.

$$\begin{align*} \underline{\Psi}_1 &= \underbrace{{\color{green}{L_{11}\underline{I}_1}}}_{\text{self-linkage of coil 1}} + \underbrace{{\color{blue}{M_{12}\underline{I}_2}}}_{\text{mutual linkage from coil 2}}, \\[4pt] \underline{\Psi}_2 &= \underbrace{{\color{blue}{M_{21}\underline{I}_1}}}_{\text{mutual linkage from coil 1}} + \underbrace{{\color{green}{L_{22}\underline{I}_2}}}_{\text{self-linkage of coil 2}}. \end{align*}$$

For most transformer calculations we use the symmetric case of the mutual inductances.
(this is true for passive, stationary, and reciprocal situations, like transformers, but not necessarily for

motors or complex setups)

$$\begin{align*} {\color{blue}{M_{12}}}={\color{blue}{M_{21}}}={\color{blue}{M}}. \end{align*}$$

Often, the self-inductances are shortened: $$\begin{align*} \color{green}{L_{11}} \rightarrow \color{green}{L_{1}} \\ \color{green}{L_{22}} \rightarrow \color{green}{L_{2}} \\ \end{align*}$$

Then

$$\begin{align*} \boxed{ \begin{pmatrix} \underline{\Psi}_1\\ \underline{\Psi}_2 \end{pmatrix} = \begin{pmatrix} {\color{green}{L_{1}}} & {\color{blue}{M}}\\ {\color{blue}{M}} & {\color{green}{L_{2}}} \end{pmatrix} \begin{pmatrix} \underline{I}_1\\ \underline{I}_2 \end{pmatrix} } \end{align*}$$

Based, on the block19 and block20 of last semester, the inducances can be calculated by the reluctance $R_{\rm mFe}$ of the iron core and the number of windings N_1, N_2:

$$\begin{align*} \boxed{ L_{1} = \frac{N_1^2}{R_{\rm mFe} } \\ L_{2} = \frac{N_2^2}{R_{\rm mFe} } \\ M = \frac{N_1 N_2}{R_{\rm mFe} } \\ } \end{align*}$$

By this, we also get:

$$\begin{align*} M=k\sqrt{L_{1}L_{2}}. \end{align*}$$

Here k is the coupling coefficient. In the shown transformer k is 1 since all flux from 1 flows through 2 and vice versa.
In reality that is not the case as explained in the next chapters.

Coupling coefficient	**Interpretation**	**Typical example**
$k=0$	no useful flux from one coil links the other coil	coils far apart
$0<k<1$	partial coupling	wireless charger with air gap or misalignment
$k\approx 1$	almost all useful flux links both coils	transformer with iron core

Tab. 1: Meaning of the coupling coefficient k

The mutual inductance M answers the question:

> How much flux linkage appears in coil 2 when the current in coil 1 changes?

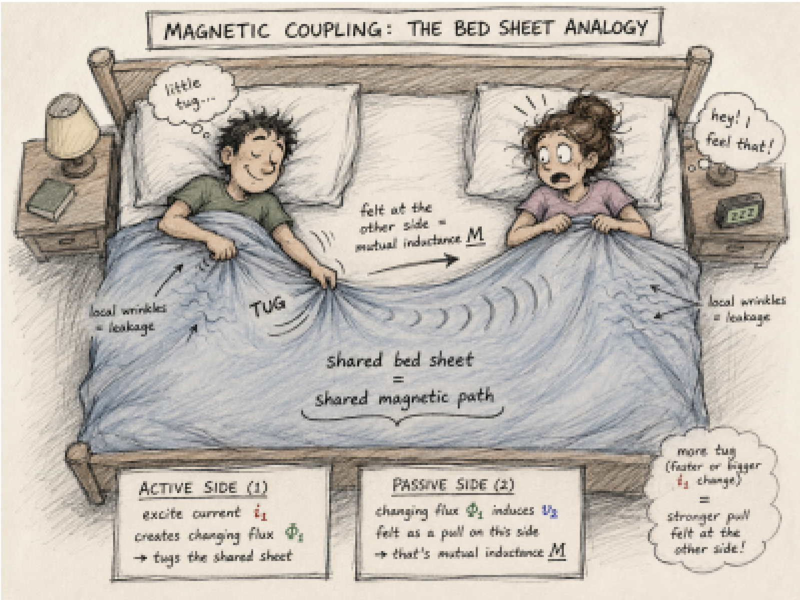
- A large M means strong interaction.
- A small M means weak interaction.

Engineering example: wireless charging

In wireless charging, the transmitter coil and receiver coil are separated by an air gap. The coupling coefficient \(k\) is much smaller than in a transformer with an iron core.

If the receiver is misaligned, less flux from the transmitter passes through it. Then \(M\) decreases, the induced voltage decreases, and the transmitted power decreases.

Analogy: shared bed sheet



Imagine two people lying on a bed holding the same bed sheet at different positions.

- \(L_{1}\):
  how strongly winding 1 couples its current into the shared magnetic path,
  like person 1 moving the sheet at their hand.
- \(L_{2}\):
  how strongly winding 2 couples its current into the shared magnetic path,
  like person 2 moving the sheet at their hand.
- \(M\):
  how strongly the motion from one hand is felt at the other hand through the same sheet.

In transformer language, \(L_{1}\) and \(L_{2}\) describe each winding's connection to the shared main flux path.
The mutual inductance \(M\) describes the transfer between the two windings through this shared path.

voltages by mutual inductances and resistances

For positive coupling, we get the following complex representation (since $u(t) = L\frac{{\rm d}i}{{\rm d}t} \: \longrightarrow \:\underline{U}=j\omega L \underline{I}$):

\[\begin{align*} \underline{U}_1 &= R_1 \underline{I}_1 + {\color{green} {j\omega L_{1} \underline{I}_1 }} + {\color{blue}{j\omega M \underline{I}_2 }}, \\[4pt] \underline{U}_2 &= R_2 \underline{I}_2 + {\color{blue} {j\omega M \underline{I}_1 }} + {\color{green}{j\omega L_{2} \underline{I}_2 }}. \end{align*} \]

For negative coupling, the sign of the \(M\)-term changes in the chosen equation system, see figure 2 and figure 3.

Fig. 2: Positive coupling: currents enter corresponding dotted terminals.

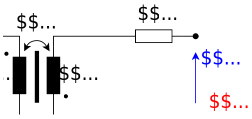
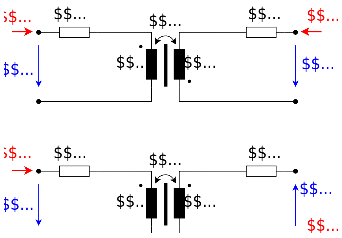


Fig. 3: Negative coupling: only one current enters a dotted terminal.



Tunnel Analogy for AC circuits

The dots are like matching openings for magnetic action.
A positive reference current (e.g. i_1) entering the dotted terminal of one winding produces a positive induced voltage (e.g. aligned with u_2) at the dotted terminal of the other winding.
With only a load R_2 connected to the secondary side, this voltage tends to drive current out of the dotted terminal into the load.

Stray and Leakage

The image of the ideal transformer shall be consecutively developed to a more realistic transformer model.
To do so, we look at the situation of two coils near each other and expand this formula for the induced voltage.

Fig. 4: Mutual induction of two coils: only part of the flux created by coil \(1\) links coil \(2\).

The flux created by coil 1 can be split into

$$\begin{align*} \Phi_{11} = {\color{blue}{\Phi_{21}}} + {\color{orange}{\Phi_{1 \rm \sigma}}}. \end{align*}$$

- Φ_{11}: total flux created by coil 1.
- ${\color{blue}{\Phi_{21}}}$: part of this flux that also links coil 2.
  This part is also ${\color{blue}{\Phi_{21}}} = \Phi_{1\rm H}$. The 'H' denotes the German word “Haupt” (sometimes also given as 'm' for “main”).
- ${\color{orange}{\Phi_{1 \rm \sigma}}}$: stray or leakage flux that does **not** link coil 2.
  The Greek letter sigma σ is used to denote leakage or “stray” quantities.

For an example, we will have a look at the instantaneous voltage induced in coil 2:

$$\begin{align*} u_{2,\rm ind}(t) = \frac{{\rm d}{\color{blue}{\Psi_{21}}}}{{\rm d}t} = N_2\frac{{\rm d}{\color{blue}{\Phi_{21}}}}{{\rm d}t}. \end{align*}$$

The complex voltage induced in coil 2 is

$$\begin{align*} \underline{U}_{2} = j\omega {\color{blue}{ \underline{\Psi}_{21} }} = j\omega N_2 {\color{blue}{ \underline{\Phi}_{21} }} . \end{align*}$$

Analogies

Analogy 1: two pendulums connected by a spring

Imagine two pendulums connected by a weak spring.

- If pendulum 1 moves, the spring can make pendulum 2 move as well.
- A strong spring transfers the motion strongly.
- A weak spring transfers the motion only weakly.
- If the spring is missing, pendulum 2 does not react.

For coupled coils:

- the changing motion corresponds to changing current,
- the spring corresponds to the magnetic coupling,
- the motion transferred to the second pendulum corresponds to the induced voltage,
- weak coupling means that only a small part of the magnetic flux links both coils.

Analogy 2: a leaky magnetic pipe

The magnetic core can be imagined as a pipe guiding magnetic flux.

- A good iron core is like a wide, low-resistance pipe: most flux reaches the second coil.
- A large air gap is like a narrow, difficult path: less flux reaches the second coil.
- Leakage flux is like flow escaping through side paths: it belongs to the first coil but does not help the second coil.

This image is helpful for transformers, wireless charging coils, and current sensors.

Engineering examples

- **Transformer:** very strong coupling because the iron core guides most of the flux through both windings.
- **Wireless charger:** weaker coupling because the flux must cross an air gap and the coils may be misaligned.
- **Current transformer:** the measured conductor acts like a one-turn primary winding; the secondary winding detects the changing magnetic field.
- **Relay coil near signal wiring:** unwanted coupling can induce noise voltages in nearby loops.

Real transformer: leakage and losses

In a real transformer, not all flux links both windings.

- The **main flux** $\Phi_{\rm H}$ links primary and secondary winding.
- The **primary leakage flux** $\Phi_{1\sigma}$ mainly links only the primary winding.
- The **secondary leakage flux** $\Phi_{2\sigma}$ mainly links only the secondary winding.

The leakage flux can be interpreted as an additional virtual branch in the mechanical setup, see figure 5.

Fig. 5: Main flux and leakage fluxes in a real transformer.

The real flux linkage equations become

$$\begin{align*} \underline{\Psi}_1 &= \underbrace{{\color{blue}{L_{1{\rm H}}\underline{I}_1+M\underline{I}_2}}}_{\text{main magnetic path}} + \underbrace{{\color{orange}{L_{1\sigma}\underline{I}_1}}}_{\text{primary leakage}}, \\[4pt] \underline{\Psi}_2 &= \underbrace{{\color{blue}{L_{2{\rm H}}\underline{I}_2+M\underline{I}_1}}}_{\text{main magnetic path}} + \underbrace{{\color{orange}{L_{2\sigma}\underline{I}_2}}}_{\text{secondary leakage}}. \end{align*}$$

Equivalently,

$$\begin{align*} \underline{\Psi}_1 &= L_1\underline{I}_1+M\underline{I}_2, & L_1&=L_{1{\rm H}}+L_{1\sigma}, \\ \underline{\Psi}_2 &= L_2\underline{I}_2+M\underline{I}_1, & L_2&=L_{2{\rm H}}+L_{2\sigma}. \end{align*}$$

The winding resistances R_1 and R_2 cause copper losses:

$$\begin{align*} P_{\rm Cu,1}=R_1I_1^2, \qquad P_{\rm Cu,2}=R_2I_2^2. \end{align*}$$

$$\begin{align*} \underline{U}_1 &= \underbrace{{\color{red}{R_1\underline{I}_1}}}_{\text{primary copper drop}} + \underbrace{{\color{orange}{j\omega L_{1\sigma}\underline{I}_1}}}_{\text{primary leakage drop}} + \underbrace{{\color{blue}{j\omega L_{1{\rm H}}\underline{I}_1+j\omega M\underline{I}_2}}}_{\text{main magnetic coupling}}, \\[6pt] \underline{U}_2 &= \underbrace{{\color{red}{R_2\underline{I}_2}}}_{\text{secondary copper drop}} + \underbrace{{\color{orange}{j\omega L_{2\sigma}\underline{I}_2}}}_{\text{secondary leakage drop}} + \underbrace{{\color{blue}{j\omega L_{2{\rm H}}\underline{I}_2+j\omega M\underline{I}_1}}}_{\text{main magnetic coupling}}. \end{align*}$$

Color scheme for the equivalent equations

In the previous formulas:

- red terms:
  winding resistance and copper loss, which convert electrical energy into heat.
- orange terms:
  leakage flux, which is unwanted but unavoidable, because magnetic field lines can also close through the surrounding air.
- blue terms:
  useful main magnetic coupling, which is responsible for transformer action.

Analogy: useful road and side roads

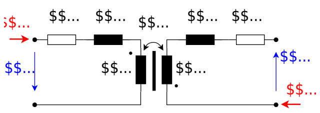
Think of the main flux as traffic on the useful road between two cities. Traffic on side roads still exists, but it does not help transport goods between the two cities.

- main flux: useful road between primary and secondary winding,
- leakage flux: side roads that return locally,
- winding resistance: friction that turns useful energy into heat.

Reduced equivalent circuit referred to the primary side

Since we know, that we can transform the current and voltage by the transformer, we can use this also to simplify the circuit.

Fig. 6: equivalent circuit of a real transformer



For calculations it is convenient to move all secondary-side quantities to the primary side.
This is called **referring** or **transforming** the secondary side to the primary side.

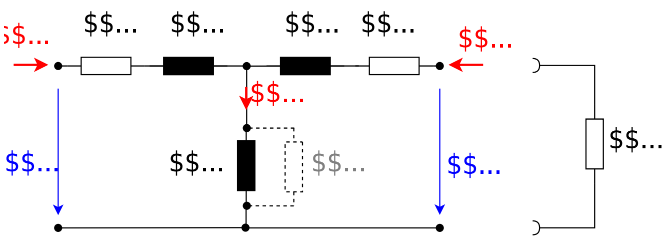
\[\begin{align*} \boxed{ \underline{U}'_2=n\underline{U}_2 } \qquad \boxed{ \underline{I}'_2=\frac{1}{n}\underline{I}_2 } \end{align*} \]

The secondary resistance and leakage reactance are transformed by \(n^2\) since $R = U / I$:

\[\begin{align*} \boxed{ R'_2=n^2R_2 } \qquad \boxed{ X'_{2\sigma}=n^2X_{2\sigma} }

\end{align*} \]

Fig. 7: reduced equivalent circuit of a real transformer



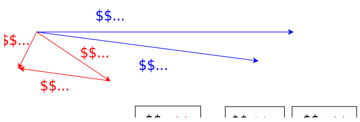
In the reduced equivalent circuit:

- \(R_1\) and \(R'_2\) model copper losses.
- \(jX_{1\sigma}\) and \(jX'_{2\sigma}\) model leakage flux.
- \(jX_{1{\rm H}}\) models the magnetizing branch.
- \(R_{\rm Fe}\) is placed parallel to \(jX_{1{\rm H}}\) to model iron losses.
- \(R_{\rm L}\) is a load resistor for the phasor diagram.

The phasor diagram is shown in figure 8.

- The upper one only shows the main voltages and currents.
  For an ohmic load current is parallel to $\underline{U}'_2$. However, $\underline{I}'_2$ is drawn as a current entering the transformer secondary port. since the transformer delivers power to the load, $\underline{I}'_2$ is antiparallel to the load current and therefore antiparallel $\underline{U}'_2$.
- The lower one shows all voltages and currents.

Fig. 8: phasor diagram of a real transformer



Why the reduced circuit is useful

Once all quantities are referred to one side, the transformer can be calculated like an AC network with impedances.
This uses the same method as complex network calculation: replace components by impedances and apply Kirchhoff's laws.

No-load operation of the real transformer

No-load operation means that the secondary side is open:

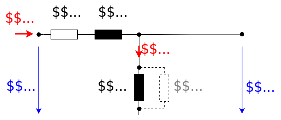
$$\underline{I}_2=0.$$

The primary side still draws a small no-load current $\underline{I}_{10}$. This current has two parts:

$$\underline{I}_{10} = \underline{I}_{\rm Fe} + \underline{I}_{\rm m}.$$

- $\underline{I}_{\rm Fe}$: current through $R_{\rm Fe}$, in phase with voltage, represents iron losses.
- $\underline{I}_{\rm m}$: magnetizing current through $jX_{1{\rm H}}$, approximately 90° lagging.

Fig. 9: No-load phasor diagram: the no-load current is the sum of iron-loss current and magnetizing current.



The technical voltage ratio is often defined from the no-load voltages. Here it is denoted by \(\ddot{u}\):

\[\boxed{ \begin{align*} \ddot{\rm u} = \frac{\text{higher voltage}}{\text{lower voltage}} \bigg|_{\rm no~load}. \end{align*} } \]

For a step-down transformer:

\[\begin{align*} \ddot{\rm u} = \frac{U_{1{\rm N}}}{U_{20}}. \end{align*} \]

Here \(U_{1{\rm N}}\) is the rated primary voltage and \(U_{20}\) is the open-circuit secondary voltage.

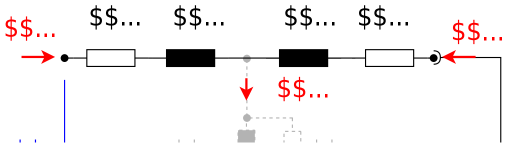
> Because of real voltage drops and magnetizing effects,
>
> \[\begin{align*} \ddot{\ rm u}\neq n, \end{align*} \]
>
> but for many practical transformers
>
> \[\begin{align*} \ddot{\rm u}\approx n. \end{align*} \]

Short-circuit operation of the real transformer

In the short-circuit test, the secondary side is shorted, so $\underline{U}_2=0$.
The primary voltage $\underline{U}_1$ is increased only until rated current flows.
Because the short-circuit impedance is small, this requires only a small fraction of the rated primary voltage.
Therefore the main flux and the magnetizing current are small, so the magnetizing branch can usually be neglected.

\[\begin{align*} | jX_{1{\rm H}} \; || \;R_{\rm Fe} | \;\; \gg \;\; | jX_{1\sigma} + \;R_1 + jX'_{2\sigma} + \;R'_2 | . \end{align*} \]

Fig. 10: Short-circuit equivalent circuit of a real transformer.



This gives the short-circuit equivalent circuit with

$$ \begin{align*} \boxed{ R_{\rm k}=R_1+R'_2 } \qquad \boxed{ X_{\rm k}=X_{1\sigma}+X'_{2\sigma} } \end{align*} $$

and

$$ \begin{align*} \underline{Z}_{\rm k} = R_{\rm k}+jX_{\rm k}. \end{align*} $$

Definition: rated short-circuit voltage

The **rated short-circuit voltage** $U_{1{\rm k}}$ is the primary voltage that must be applied while the secondary side is shorted so that rated primary current $I_{1{\rm N}}$ flows.

As a relative value:

$$ \begin{align*} \boxed{ {\rm u_k} = \frac{U_{1{\rm k}}}{U_{1{\rm N}}}\cdot 100~\% } \end{align*} $$

- Small ${\rm u_k}$ means: small internal impedance.
  When a short-circuit fault on the secondary side happens, the input current can get very high.
- Large ${\rm u_k}$ means: high internal impedance.
  Stronger current limitation, but also larger voltage drop under load.

The continuous short-circuit current for rated primary voltage is

$$\begin{align*} \boxed{ I_{1{\rm k}} = \frac{U_{1{\rm N}}}{U_{1{\rm k}}}\cdot I_{1{\rm N}} = I_{1{\rm N}}\cdot \frac{100~\%}{{\rm u_k}} } \end{align*}$$

where $u_{\rm k}$ is inserted as a percentage value.

Definition: approximation for the first peak current

For a first approximation of the maximum instantaneous current the following formula can be used:

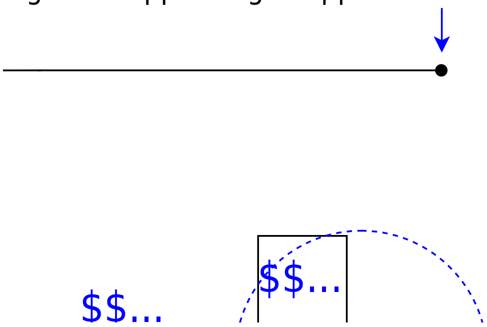
$$\begin{align*} \boxed{ i_{{\rm peak}} = 2.54 \cdot I_{1 \rm k} } \end{align*}$$

Real transformer under load

Under load, the short-circuit equivalent circuit is often sufficient for engineering estimates.

$$\begin{align*} \underline{U}_{\rm k} = \left(R_{\rm k}+jX_{\rm k}\right)\underline{I}_1. \end{align*}$$

Fig. 11: Kapp triangle: approximate voltage drop under load using R_k*I and X_k*I.



This voltage drop is subtracted vectorially from the primary-side voltage relation.

$$\begin{align*} \boxed{ \underline{U_1} - \underline{U}_{\rm k} = \underline{U'_2} = n \underline{U_2} } \end{align*}$$

If the magnetizing branch is neglected for the load calculation, then the series current relation is approximately $\underline{I}'_2 \approx -\underline{I}_1$. Therefore the ideal current transformation can still be used as an approximation.

$$\begin{align*} \boxed{ \frac{\underline{I}_1}{\underline{I}_2} = -\frac{N_2}{N_1} = -\frac{1}{n} } \end{align*}$$

The Kapp triangle (see yellow triangle in figure 11) represents the formula for $\underline{U}_{\rm k}$.

Fig. 12: animation of Kapp triangle

press here for the animation

Engineering use: voltage regulation

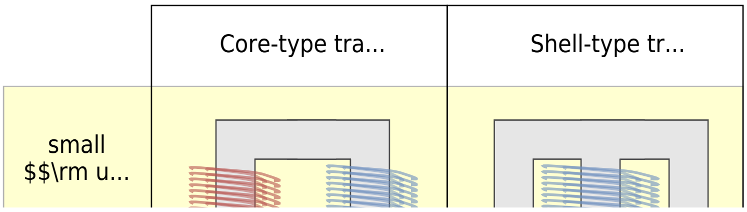
In a robot with motors, the supply transformer may show a lower output voltage during high acceleration because the motor currents increase. The Kapp triangle helps estimate this voltage drop. This is important for:

- selecting transformer size,
- checking whether the DC link after a rectifier remains high enough,
- designing fuses and protective devices,
- avoiding undervoltage resets in control electronics.

Construction types and cooling

Transformer behavior is influenced by construction.

Fig. 13: Transformer types



Cooling types:

- **Dry-type transformer:** air cooling, often used inside machines or buildings at lower and medium power.
- **Oil transformer:** oil provides insulation and heat transfer, typical for higher power.

Mechatronics examples

- **Isolating transformer:** safe diagnostic supply for laboratory setups.
- **Control transformer:** supplies $24~{\rm V}$ or similar low-voltage control circuits.
- **Current transformer:** measures large motor currents with galvanic isolation.
- **Welding transformer:** intentionally high short-circuit voltage and current limitation for welding processes.

Typical technical transformer data

Name / use	**Typical ${\rm u_k}$**	**Secondary voltage U_2**	**Important note**
Power transformer	$4\ldots 12~\%$	application-dependent	low voltage drop, high fault currents possible
Isolating transformer	$\approx 10~\%$	max. $250~{\rm V}$	galvanic isolation for safety and measurement
Toy transformer	$\approx 20~\%$	max. $24~{\rm V}$	current limitation is desired
Doorbell transformer	$\approx 40~\%$	max. $12~{\rm V}$, often several taps	simple robust low-voltage supply
Ignition transformer	$\approx 100~\%$	$\leq 14~{\rm kV}$	high voltage, limited current
Welding transformer	$\approx 100~\%$	max. $70~{\rm V}$	large current, strong current limitation
Voltage transformer	$<1~\%$	$100~{\rm V}$	operate with high load resistance, approximately no-load
Current transformer	$100~\%$	$0~{\rm V}$ ideal secondary voltage	operate with low burden, approximately short-circuit

Tab. 2: Typical transformer types, short-circuit voltage, and secondary voltage

Important safety note: current transformers

A current transformer secondary must not be opened while primary current flows.
If the secondary circuit is open, the transformer tries to maintain the magnetic balance and can generate dangerous high voltages.

Exercises

Exercise E2.1 Quick check: ideal transformer voltage and current ratio

A transformer has $N_1=1200$ turns and $N_2=300$ turns. The primary RMS voltage is $U_1=230~{\rm V}$. The secondary side supplies a load current $I_2=2.0~{\rm A}$.

- Calculate the turns ratio n.
- Calculate the ideal secondary voltage U_2.
- Calculate the magnitude of the ideal primary current I_1.
- State whether this is a step-up or step-down transformer.

Result

$$\begin{align*} n=\frac{N_1}{N_2} = \frac{1200}{300} = 4. \end{align*}$$

The secondary voltage is

$$\begin{align*} U_2 = \frac{U_1}{n} = \frac{230~{\rm V}}{4} = 57.5~{\rm V}. \end{align*}$$

The primary current magnitude is

$$\begin{align*} I_1 = \frac{I_2}{n} = \frac{2.0~{\rm A}}{4} = 0.50~{\rm A}. \end{align*}$$

Because $U_2<U_1$, it is a step-down transformer.

Exercise E3.1 Quick check: mutual inductance from reluctance

Two coils are wound on the same ideal magnetic core. The main magnetic reluctance is

$$\begin{align*} R_{\rm mH}=2.0\cdot 10^6~\frac{1}{\rm H}. \end{align*}$$

The number of turns is $N_1=500$ and $N_2=100$.

- Calculate $L_{1{\rm H}}$.
- Calculate $L_{2{\rm H}}$.
- Calculate M.
- Check whether the units are correct.

Result

$$\begin{align*} L_{1{\rm H}} = \frac{N_1^2}{R_{\rm mH}} = \frac{500^2}{2.0\cdot 10^6~1/{\rm H}} = 0.125~{\rm H}. \end{align*}$$

$$\begin{align*} L_{2{\rm H}} = \frac{N_2^2}{R_{\rm mH}} = \frac{100^2}{2.0\cdot 10^6~1/{\rm H}} = 0.0050~{\rm H} = 5.0~{\rm mH}. \end{align*}$$

$$\begin{align*} M = \frac{N_1N_2}{R_{\rm mH}} = \frac{500\cdot 100}{2.0\cdot 10^6~1/{\rm H}} = 0.025~{\rm H} = 25~{\rm mH}. \end{align*}$$

The unit is correct because $1/(1/{\rm H})={\rm H}$.

Exercise E4.1 Quick check: referring secondary quantities to the primary side

A transformer has $n=5$. The secondary winding resistance is $R_2=0.20~\Omega$ and the secondary leakage reactance is $X_{2\sigma}=0.35~\Omega$.

Calculate the values R'_2 and $X'_{2\sigma}$ referred to the primary side.

Result

$$\begin{align*} R'_2 = n^2R_2 = 5^2\cdot 0.20~\Omega = 25\cdot 0.20~\Omega = 5.0~\Omega. \end{align*}$$

$$\begin{align*} X'_{2\sigma} = n^2X_{2\sigma} = 5^2\cdot 0.35~\Omega = 25\cdot 0.35~\Omega = 8.75~\Omega. \end{align*}$$

The unit remains Ω, because n is dimensionless.

Exercise E5.1 Quick check: short-circuit voltage and fault current

A transformer has a rated primary current $I_{1{\rm N}}=10~{\rm A}$ and a short-circuit voltage $u_{\rm k}=5~\%$.

- Calculate the continuous short-circuit current $I_{1{\rm k}}$ when rated primary voltage is applied.
- Estimate the initial peak short-circuit current $i_{\rm p}$ using $i_{\rm p}\approx 2.54 I_{1{\rm k}}$.

Result

$$\begin{align*} I_{1{\rm k}} = I_{1{\rm N}}\cdot \frac{100~\%}{u_{\rm k}} = 10~{\rm A}\cdot \frac{100~\%}{5~\%} = 200~{\rm A}. \end{align*}$$

$$\begin{align*} i_{\rm p} \approx 2.54\cdot I_{1{\rm k}} = 2.54\cdot 200~{\rm A} = 508~{\rm A}. \end{align*}$$

The short-circuit current is much larger than the rated current. Protection devices must be selected accordingly.

Exercise E6.1 Longer exercise: transformer equivalent circuit for an actuator supply

A single-phase transformer supplies an actuator driver. Rated data and equivalent circuit data are:

$$\begin{align*} U_{1{\rm N}}&=230~{\rm V}, & U_{2{\rm N}}&=23~{\rm V}, & I_{2{\rm N}}&=5.0~{\rm A}, \\ R_1&=1.2~\Omega, & X_{1\sigma}&=1.8~\Omega, \\ R_2&=0.012~\Omega, & X_{2\sigma}&=0.018~\Omega. \end{align*}$$

Assume $n=\frac{U_{1{\rm N}}}{U_{2{\rm N}}}$. The magnetizing branch is neglected for the loaded operating point.

- Calculate n.
- Refer R_2 and $X_{2\sigma}$ to the primary side.
- Calculate $R_{\rm k}$ and $X_{\rm k}$.
- Calculate the primary rated current magnitude $I_{1{\rm N}}$ using the ideal current ratio.
- Estimate the magnitude of the internal voltage drop $U_{\rm k}\approx |\underline{Z}_{\rm k}|I_{1{\rm N}}$.

Result

The turns ratio is

$$\begin{align*} n = \frac{U_{1{\rm N}}}{U_{2{\rm N}}} = \frac{230~{\rm V}}{23~{\rm V}} = 10. \end{align*}$$

The secondary quantities referred to the primary side are

$$\begin{align*} R'_2 &= n^2R_2 = 10^2\cdot 0.012~\Omega = 1.2~\Omega, \\ X'_{2\sigma} &= n^2X_{2\sigma} = 10^2\cdot 0.018~\Omega = 1.8~\Omega. \end{align*}$$

Therefore

$$\begin{align*} R_{\rm k} &= R_1+R'_2 = 1.2~\Omega+1.2~\Omega = 2.4~\Omega, \\ X_{\rm k} &= X_{1\sigma}+X'_{2\sigma} = 1.8~\Omega+1.8~\Omega = 3.6~\Omega. \end{align*}$$

The primary current magnitude is

$$\begin{align*} I_{1{\rm N}} = \frac{I_{2{\rm N}}}{n} = \frac{5.0~{\rm A}}{10} = 0.50~{\rm A}. \end{align*}$$

The magnitude of the short-circuit impedance is

$$\begin{align*} |\underline{Z}_{\rm k}| = \sqrt{R_{\rm k}^2+X_{\rm k}^2} = \sqrt{(2.4~\Omega)^2+(3.6~\Omega)^2} = 4.33~\Omega. \end{align*}$$

Thus the internal voltage drop estimate is

$$\begin{align*} U_{\rm k} \approx |\underline{Z}_{\rm k}|I_{1{\rm N}} = 4.33~\Omega\cdot 0.50~{\rm A} = 2.17~{\rm V}. \end{align*}$$

This is a primary-side voltage drop. On the secondary side it corresponds approximately to

$$\begin{align*} \frac{2.17~{\rm V}}{10}=0.217~{\rm V}. \end{align*}$$

For a $23~{\rm V}$ actuator supply this is small but not zero.

Common pitfalls

- **Using a transformer with DC:** A transformer needs changing flux. With DC, after the switching transient, an ideal transformer no longer transfers voltage. A real transformer may overheat because the winding resistance limits the current only weakly.
- **Forgetting the current ratio sign:** The minus sign in $\frac{\underline{I}_1}{\underline{I}_2}=-\frac{1}{n}$ comes from reference arrows. Do not interpret it as negative power loss.
- **Mixing peak values and RMS values:** In AC power and transformer ratings, U and I usually mean RMS values. Time functions are written $u(t)$, $i(t)$. Instantaneous short-circuit peaks are written here as $i_{\rm p}$.
- **Confusing reluctance and resistance:** Magnetic reluctance $R_{\rm m}$ has the unit $1/{\rm H}$, not Ω.
- **Confusing n and the technical no-load voltage ratio $\ddot{u}$:** The ideal ratio is $n=\frac{N_1}{N_2}$. The measured no-load voltage ratio is close to n, but not exactly equal for a real transformer.
- **Forgetting the square when referring impedances:** Voltages transform with n, currents

with $\frac{1}{n}$, but impedances transform with n^2.
- **Ignoring leakage reactance:** Leakage reactance is often the dominant part of short-circuit impedance. It strongly affects fault current and voltage drop.
- **Treating $u_{\rm k}$ as a voltage in volts:** $u_{\rm k}$ is normally given in percent. Insert it consistently in formulas.
- **Using 2.54 as a universal law:** The first short-circuit peak depends on the R/X ratio and on the switching instant. The factor 2.54 is an approximation.
- **Opening a current transformer secondary:** This can create dangerous voltages. Current transformers are operated with a low burden, approximately as a short-circuit.
- **Assuming ideal isolation at every frequency:** Real transformers have parasitic capacitances between windings. For high-frequency noise and EMC, the “isolated” sides can still be capacitively coupled.

Embedded resources