3 Combinatorial Logic

Student Group

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3. Combinatorical Logic

introductional example

Fig. 1: Simulation of a 7-segment encoder and display

The combinatorial logic shown in <impref pic1> enables to output distinct logic values for eacht logic input. When you change the `input nibble` you can see that the the correct number appears on the 7-segment-display. By clicking onto the bits of the input nibble, you can change the number.

Tasks:

1. Which output Y_0 … Y_6 is generated from the input nibble `1000`? Which from `1001`?
2. Is the output only depending on the input? Is there a dependance on the histroy?

3.1 Combinatorical Circuit

Up to now, we looked onto simple logic circuits. Thes are relatively easy to analyze and synthesize (=develop). The main question in this chapter is: how can we set up and optimize logic circuits?

In the following we have a look onto combinatorical circuits. These are generally logic circuits with

- n inputs X_0, X_1, … X_{n-1}
- m outputs Y_0, Y_1, … Y_{m-1}
- no “memory”, that is: a given set of input bits results in a distinct output

They can be description by

- truth table
- boolean formula
- hardware description language

The ladder one is not in the focus of this course.

The applications range:

- (simple) half/full adder
- digital comparators (logic circuit to compare 2 values)
- Multiplexer / demultiplexer
- Arithmetic logic units in microcontrollers and processors
- much more

3.1.1 Example

In order to understand the synthesis of a combinatoric logic we will follow a step-by-step example for this chapter.

Imagine you are working for a company called “mechatronics and robotics”. One costumer wants to have an intelligent switch as input device connected to a microcontroller for controlling an oven. For this project “Therm-o-Safety” he needs a combinatoric logic:

- The intelligent switch has 4 user selectable positions: 1, 2, 3, 4
- Additionally there are 2 non-selectable positions for the case of failure.
- The output $Y=1$ will activate a temperature monitoring.
- The temperature monitoring has to be active for 3 and 4 and in case of a major failure. A major failure is for example, when the switch position is unclear. In this case the input of the combinatorial circuit is “ON”.
- There are no other cases of inputs.

This requirements are put into a truth table:

Therm-o-Safety				
Input	**X2**	**X1**	**X0**	**Y**
	0	0	0	-
1	0	0	1	0
2	0	1	0	0
3	0	1	1	1
4	1	0	0	1
	1	0	1	-
OFF	1	1	0	0
ON	1	1	1	1

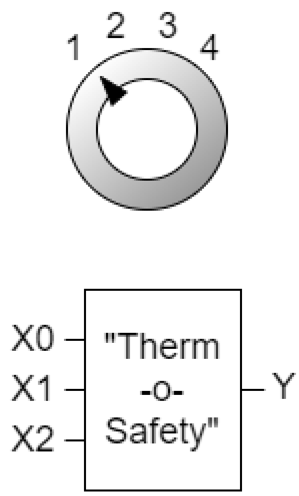


Fig. 2: Therm-o-Safety truth table

figure 2 shows one implementation of this requirements. The inputs `001` … `011` represent the inputs 1…4. The cases of failure are coded with `110` and `111`.
The output Y is activated as requested. For the two combinations `000` and `101` there is no output value defined. Depending on the requierements for a project these shall either better be `0` or `1` or the output of these does not matter. We had this “does not matter” before: the technical term is “I don't care”, and it is written as a `-` or a `x`.

By this, we have done the first step in order to syntesize the requested logic.

3.1.2 Sum of Products

Now, we want to investigate some of the input combinations (= lines in the truth table). At first, we have a look onto the input combination `011`, where the output has to be $Y=1$.

If this input combination would be the only one for the output of $Y=1$, the following could be stated: “$Y=1$ (only) when the X_0 is 1 AND X_1 is 1 AND X_2 is 0 ”. It can also be re-arranged to:

“$Y=1$ (only) when the X_0 is 1 AND X_1 is 1 AND X_2 is not 1 ”.

This statement is similar to $X_0 \cdot X_1 \cdot \overline{X_2}$. The used conjuntion resuts only in 1, when all inputs are 1. The negation of X_2 takes account of the fact, that X_2 has to be 0.

Fig. 3: Therm-o-Safety truth table - first analysis

Therm-o-Safety				
Input	**X2**	**X1**	**X0**	**Y**
	0	0	0	-
1	0	0	1	0
2	0	1	0	0
3	0	1	1	1
4	1	0	0	1
	1	0	1	-
OFF	1	1	0	0
ON	1	1	1	1

$\overline{X2}$ & X1 & X0

figure 3 shows the boolean expression for ths combination. In figure 4, this boolean expression is converted into a struction with logic gates.

Fig. 4: logic circuit for the combination '100'

With the same idea in mind, we can have a look for the other combinations resulting in $Y=1$. These are the combinations `100` and `111`:

- For `100` The statement would be: “$Y=1$ (only) when the X_0 is 0 AND X_1 is 0 AND X_2 is 1”. Similary to the combination above this leads to: $\overline{X_0} \cdot \overline{X_1} \cdot {X_2}$.
- For `111`, the boolean expression is ${X_0} \cdot {X_1} \cdot {X_2}$.

Note!

- Each row in a truth table (=one distinct combination) can be represented by a **minterm** or **maxterm**
- A **minterm** is the conjunction (AND'ing) of all inputs, where unter certain instances a negation have to be used
- In a minterm an input variable with `0` has to be negated, in order to use it as an input for the AND.
  e.g. $X_0 = 0$ AND $X_1 = 1 \quad \rightarrow \quad \overline{X_0} \cdot X_1$

Fig. 5: Therm-o-Safety truth table - sum of products

Therm-o-Safety					
Input	**X2**	**X1**	**X0**	**Y**	**minterm**
	0	0	0	-	
1	0	0	1	0	
2	0	1	0	0	
3	0	1	1	1	$\overline{X_2} \cdot X_1 \cdot X_0$
4	1	0	0	1	$X_2 \cdot \overline{X_1} \cdot \overline{X_0}$
	1	0	1	-	
OFF	1	1	0	0	
ON	1	1	1	1	$X_2 \cdot X_1 \cdot X_0$

In figure 5 all minterms for $Y=1$ are shown. The figure 6 depicts all the logic circuits for the three minterms. These lead to the outputs Y', Y'', and Y'''.

Fig. 6: logic circuit for the combinations '100', '110', '111'

For the final step we have to combine the single results for the minterms. The output has to be 1 when at least one of the minterms is 1. Therefore, the minterms have to be connected disjunctive:

\begin{align*} Y &= & Y' & \quad + & Y'' & \quad + & Y''' \\ Y &= & (X_0 \cdot X_1 \cdot \overline{X_2}) & \quad + & (\overline{X_0} \cdot \overline{X_1} \cdot {X_2}) & \quad + & ({X_0} \cdot {X_1} \cdot {X_2}) \\ \end{align*}

This leads to the logic circuit shown in figure 7. Here, you can input the different combinations by clicking onto the bits of the input nibble.

Fig. 7: logic circuit for therm-o-safety

Note!

- The disjunction of the minterms is called **sum of products**, **SoP**, **disjunctive normal form** or **DNF**.
- When all inputs are used in each of the minterms the normal form is called **full disjunctive normal form**
- When snytesizing a logic circuit by sum of procucts, all 'don't care' terms outputing 0.

We have seen, that the sum of products is one tool to derive a logic circuit based on a truth table. Alternatively it is also possible to insert an intermediate step, where the logic formula is simplified.

In the following one possible optimization is shown:

\begin{align*} Y &= & (X_0 \cdot X_1 \cdot \overline{X_2}) & \quad + & (\overline{X_0} \cdot \overline{X_1} \cdot {X_2}) & \quad + & ({X_0} \cdot {X_1} \cdot {X_2}) & \quad | \text{associative law} \\ Y &= & (\overline{X_0} \cdot \overline{X_1} \cdot {X_2}) & \quad + & (X_0 \cdot X_1 \cdot \overline{X_2}) & \quad + & ({X_0} \cdot {X_1} \cdot {X_2}) & \quad | \text{associative law } \\ Y &= & (\overline{X_0} \cdot \overline{X_1} \cdot {X_2}) & \quad + & ((X_0 \cdot X_1) \cdot \overline{X_2}) & \quad + & (({X_0} \cdot {X_1}) \cdot {X_2}) & \quad | \text{distributive law } \\ Y &= & (\overline{X_0} \cdot \overline{X_1} \cdot {X_2}) & \quad + & ((X_0 \cdot X_1) \cdot (\overline{X_2} + {X_2})) & & & \quad | \text{complementary element} \\ Y &= & (\overline{X_0} \cdot \overline{X_1} \cdot {X_2}) & \quad + & (X_0 \cdot X_1) \\ \end{align*}

3.1.3 Product of Sums

In the sub-chapter before we had a look onto the combinations which generates an output of $Y=1$ by means of the AND operator. Now we are investigating the combinations with $Y=0$. Therefore, we need an operator, which results in 0 for only on distinct combination.

The first combination to look for is `001`. If this input combination would be the only one for the output of $Y=0$, the following could be stated:
“$Y=0$ (only) when the X_0 is 1 AND X_1 is 0 AND X_2 is 0 ”.
With having the duality in mind (see cpt. The Set of Rules) the opposite is also true:
“$Y=1$ when X_0 is 0 OR X_1 is 1 OR X_2 is 1 ”

This is the same like: $\overline{X_0} + X_1 + X_2$
The booleand operator we need hiere is the OR-operator.

Simmilarly, the combinations `010` und `110` can be transformed. Keep in mind, that this time we are looking for a formula with results in 0 only for the given one distinct combination.

Note!

- A **maxterm** is the disjunction (OR'ing) of all inputs, where unter certain instances a negation have to be used.
- In a maxterm an input variable with 1 has to be negated, in order to use it as an input for the OR.

The figure 8 shows all the maxterms for the Therm-o-Safety example.

Fig. 8: Therm-o-Safety truth table

Therm-o-Safety						
Input	X2	X1	X0	Y	minterm	maxterm
	0	0	0	-		
1	0	0	1	0		$X_2 + X_1 + \overline{X_0}$
2	0	1	0	0		$X_2 + \overline{X_1} + X_0$
3	0	1	1	1	$\overline{X_2} \cdot X_1 \cdot X_0$	
4	1	0	0	1	$X_2 \cdot \overline{X_1} \cdot \overline{X_0}$	
	1	0	1	-		
OFF	1	1	0	0		$\overline{X_2} + \overline{X_1} + X_0$
ON	1	1	1	1	$X_2 \cdot X_1 \cdot X_0$	

The formulas of figure 8 can again be transformed into gate circiuts (figure 9). Here, only for the inputs '001', '010', '110' one of the outputs Y', Y'' or Y''' is 0.

Fig. 9: logic circuit for the combinations '001', '010', '110'

When these intermediate outputs Y', Y'', Y''' are used as an input for an AND-gate the resultin output will get 0 when at least one of the intermediate outputs are 0. This results in another way to synthesize the Therm-o-Safety (see figure 10)

Fig. 10: logic circuit for therm-o-safety

Also the products of sum can be simplified:

\begin{align*} Y &= & (\overline{X_0} + X_1 + X_2) & \quad \cdot & ({X_0} + \overline{X_1} + X_2) & \quad \cdot & (\overline{X_0} + \overline{X_1} + X_2) \\ &= ... \\ Y &= (\overline{X_0} + X_1 + X_2) & \quad \cdot & (\overline{X_0} + \overline{X_1}) \\ \end{align*}

This result Y by the sum of products is different compared to the result in product of sums:

- product of sums: $Y = (\overline{X_0} \cdot \overline{X_1} \cdot {X_2}) + (X_0 \cdot X_1)$
- sum of products: $Y = (\overline{X_0} + X_1 + X_2) \cdot (\overline{X_0} + \overline{X_1})$

In this case these resuls cannot be transformed into each other with the means of boolean rules.

Note!

- The disjunction of the maxterms is called **products of sum**, **PoS**, **conjunctive normal form** or **CNF**.
- When all inputs are used in each of the minterms the normal form is called **full conjunctive normal form**

- When snytesizing a logic circuit by sum of procucts, all 'don't care' terms outputing 1
- The products of sum is the DeMorgan dual of the sum of products **<u>if</u>** there are no don't care terms. Otherwise the resuls cannot be transformed into each other with the means of boolean rules.

3.2 Karnaugh Map

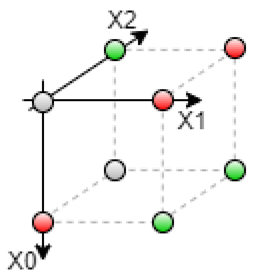
3.2.1 Introduction with example

For our therm-o-safety example we found two possible gate logics which can produce the required output. We also have seen, that optimizing the terms (i.e. the min- or maxterms) is often possible. But ip to now we do not know how we can find optimum implementation.

For this, we try to interpret the inputs of our example as dimensions in a multidimensional space. in order to do so, we have first to think aobut the dimensions

Fig. 12: Therm-o-Safety in multi-dimensional space

Therm-o-Safety				
Input	**X2**	**X1**	**X0**	**Y**
	0	0	0	-
1	0	0	1	0
2	0	1	0	0
3	0	1	1	1
4	1	0	0	1
	1	0	1	-
OFF	1	1	0	0
ON	1	1	1	1



interactive example

Exercise 3.1.x Further Querstions

1. compare the results with the output given here (the output y can be changed by clicking onto it)